BAYESIAN STATISTICS AND FUZZY INFORMATION

Reinhard Viertl

Technische Universität Wien

www.statistik.tuwien.ac.at/public/viertl/

FUZZY INFORMATION

- Fuzzy Data
- Fuzzy a-priori Knowledge
- Fuzzy Probabilities
- Soft Computing ECSC

KINDS OF DATA UNCERTAINTY

Variability

Errors

Missing Values

Imprecision (Fuzzy Data)

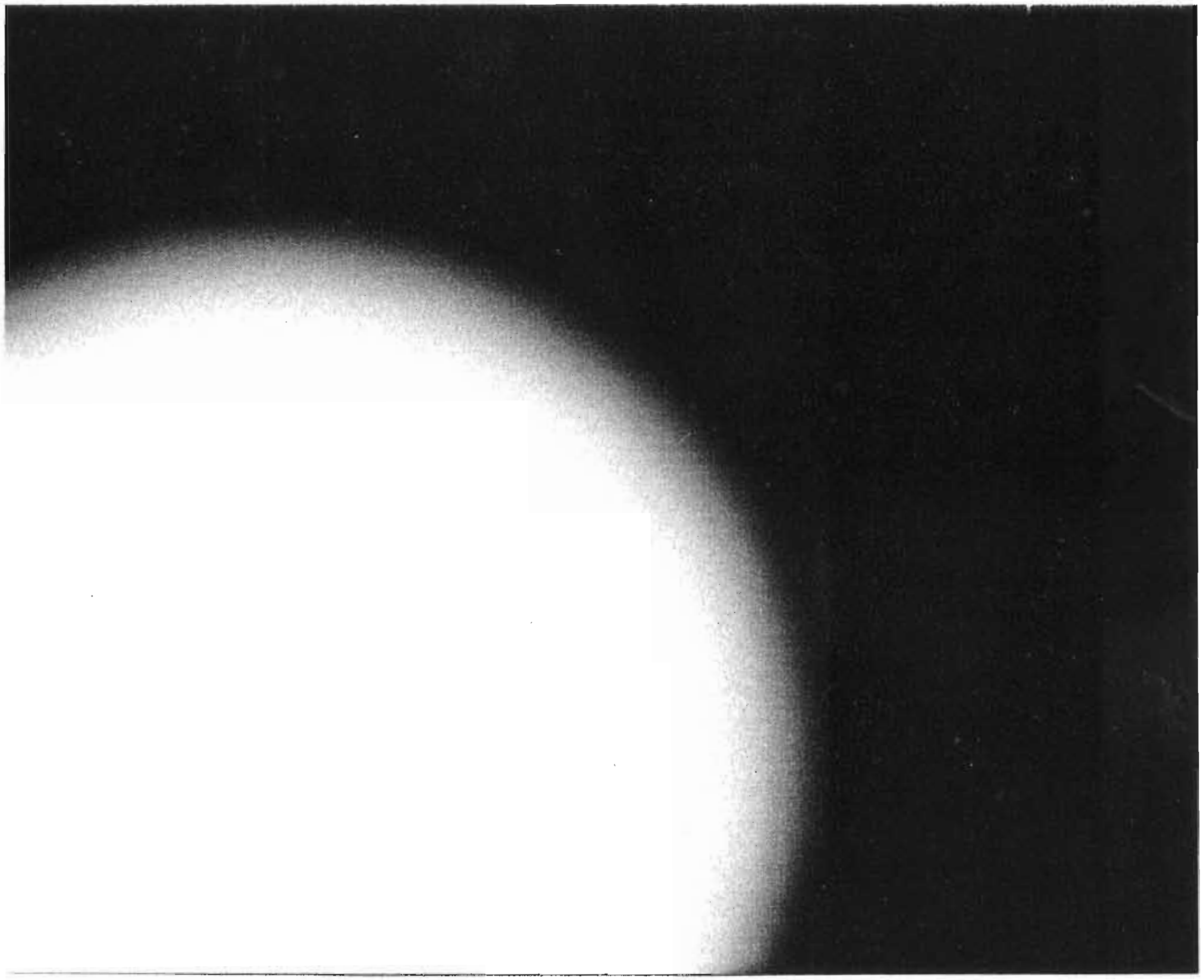
Description: Fuzzy Numbers
Fuzzy Vectors
Fuzzy Functions

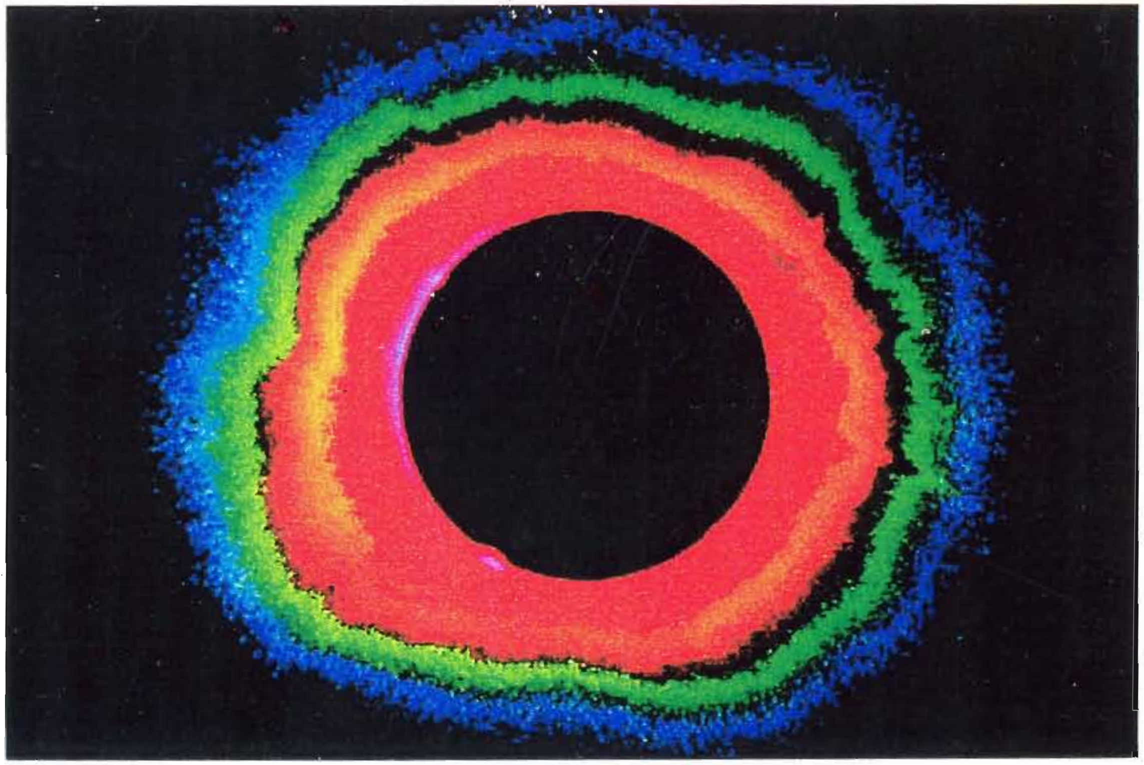
FUZZY DATA

- Environmental Data
- Recovering Times
- Quality of Life Data
- Migration Data

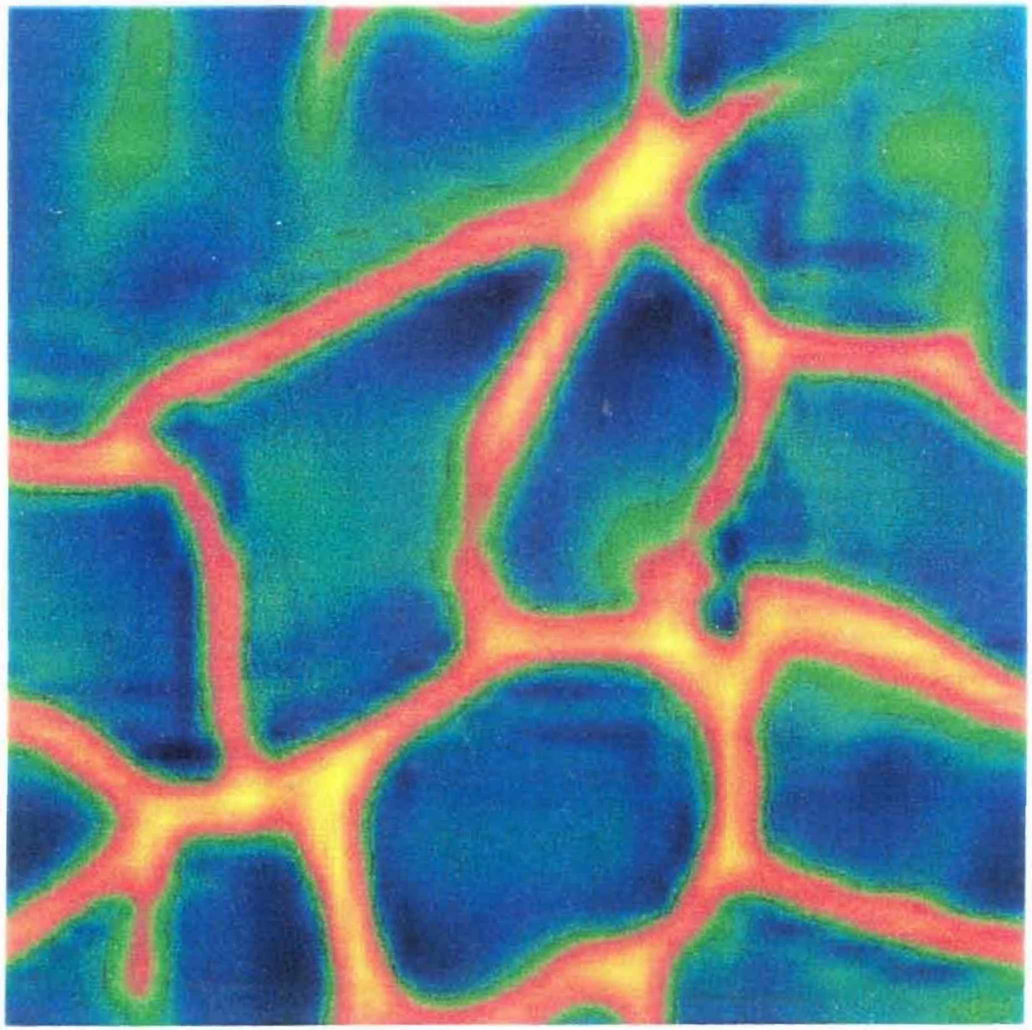
⋮

- Precision Measurement Data





False colour map of
solar corona,
showing contours of
equal brightness.



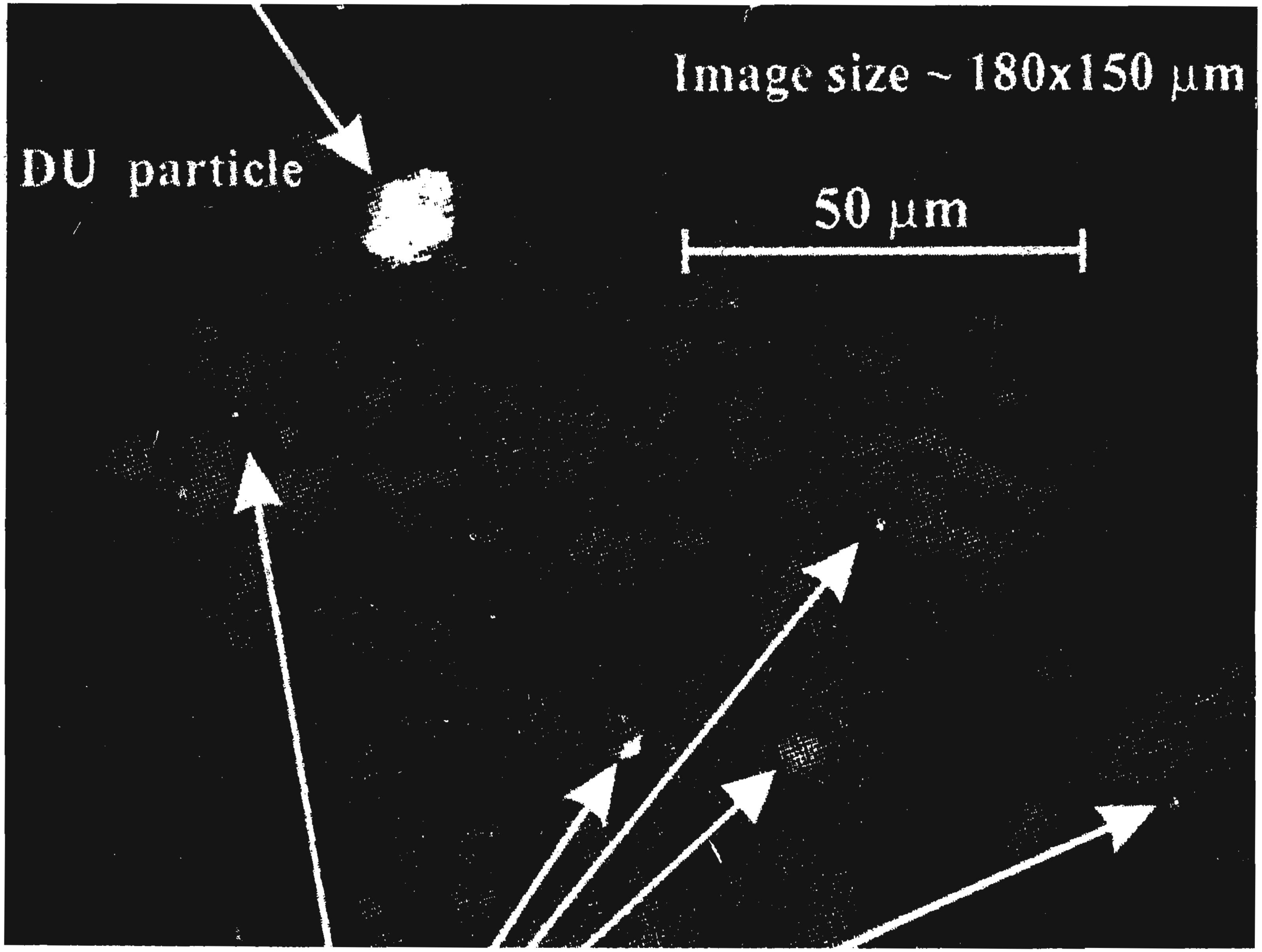
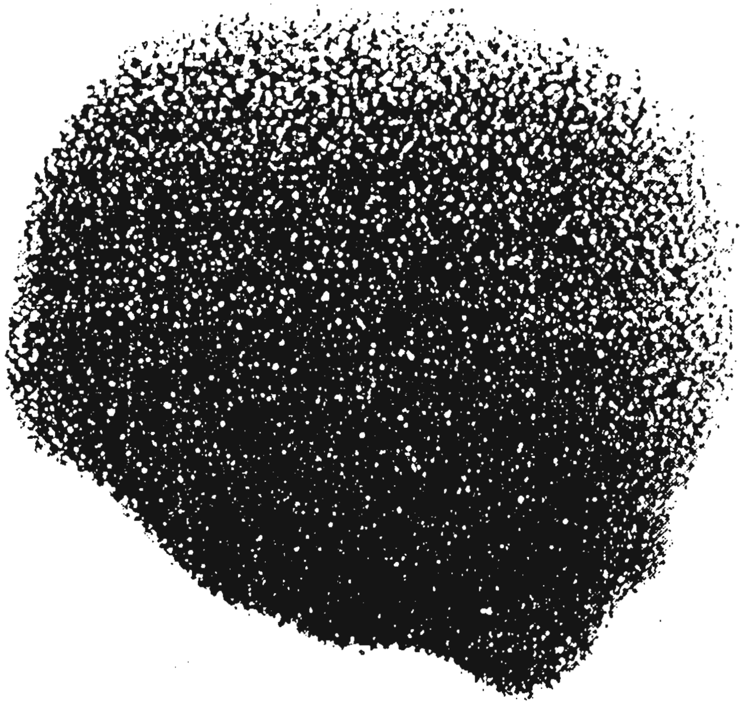
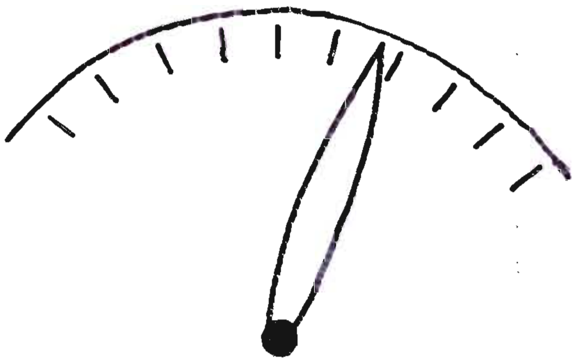


Image size ~ 180x150 μm
DU particle
50 μm



MEASUREMENTS



4.823

analog

digital

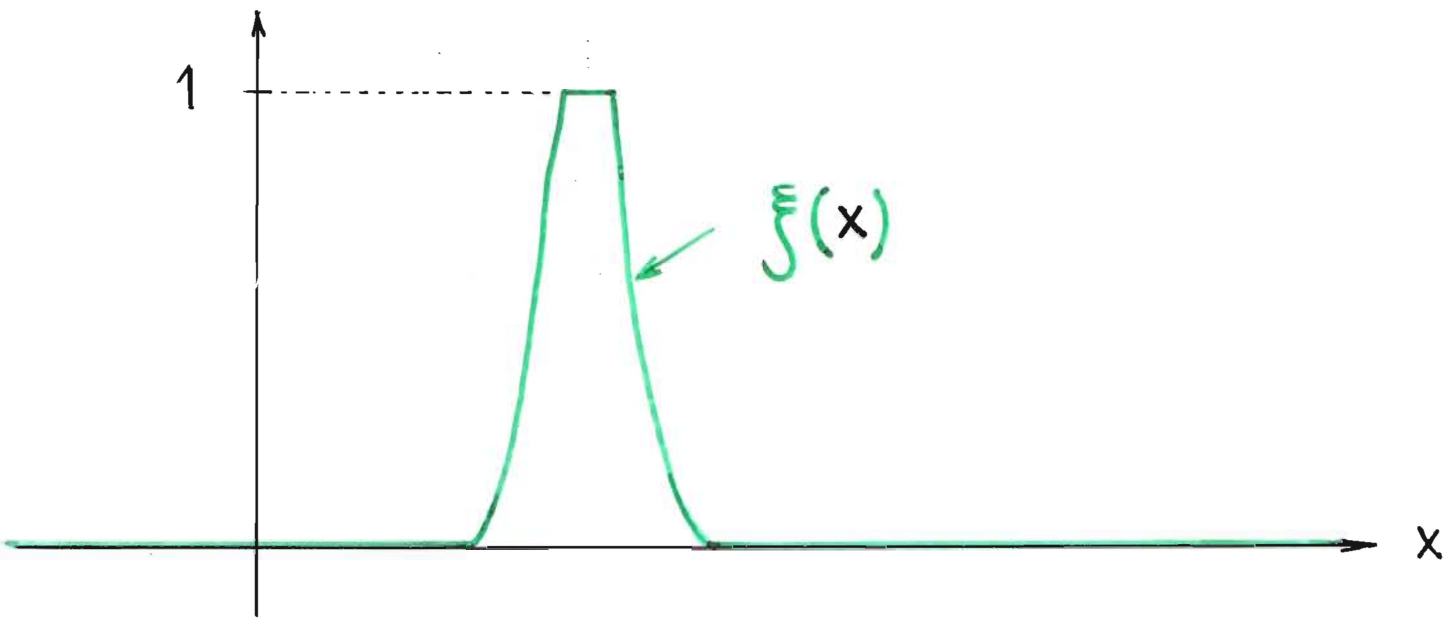
Results ?

MEASUREMENT RESULTS

Not precise numbers but more or less non-precise

Mathematical model: Fuzzy number x^*

Characterizing function $\xi(\cdot)$



Characterizing Function $\xi(\cdot)$

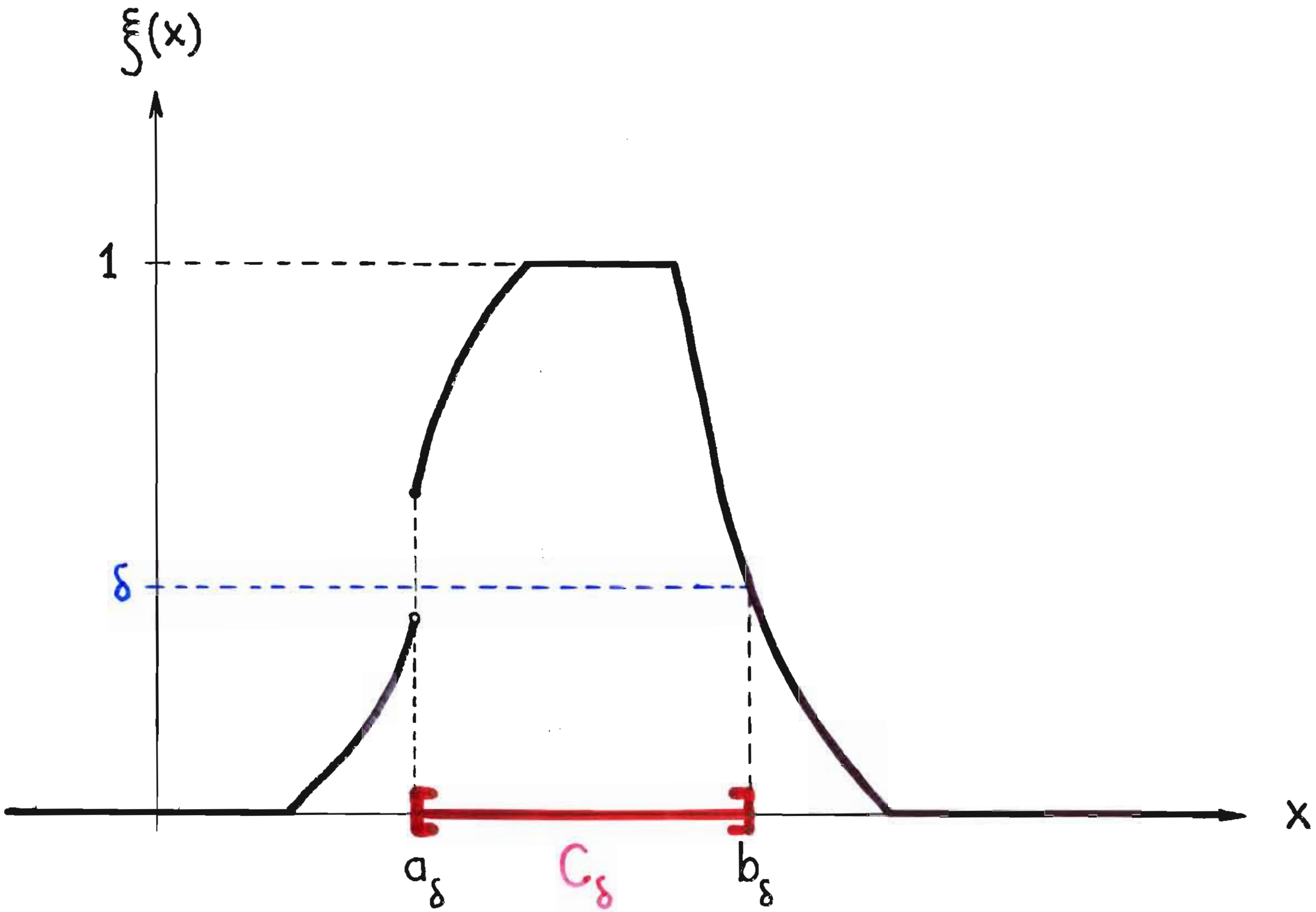
(1) $0 \leq \xi(x) \leq 1 \quad \forall x \in \mathbb{R}$

(2) support $[\xi(\cdot)]$ is bounded

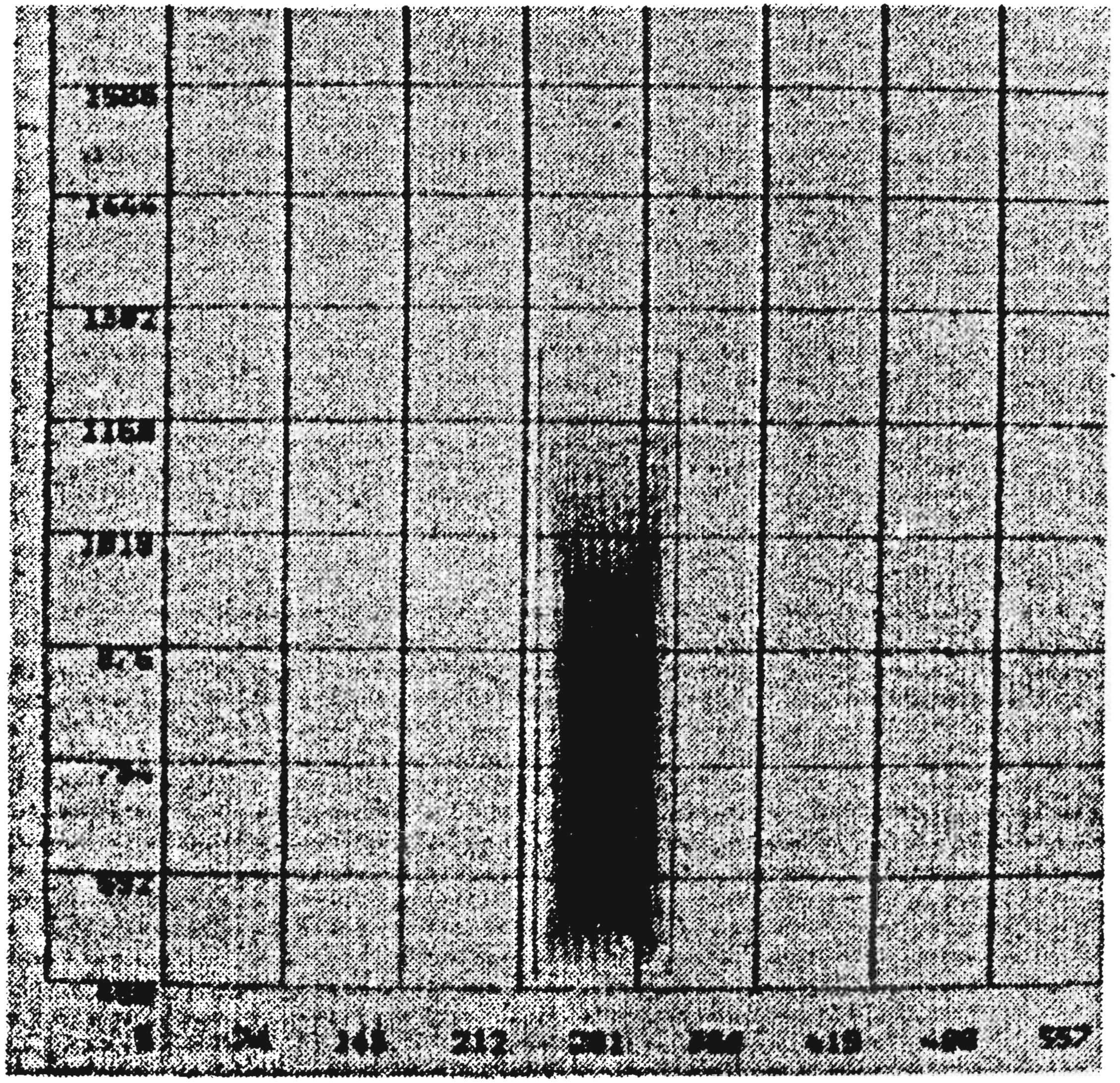
(3) $\forall \delta \in (0;1]$ the δ-Cut C_δ

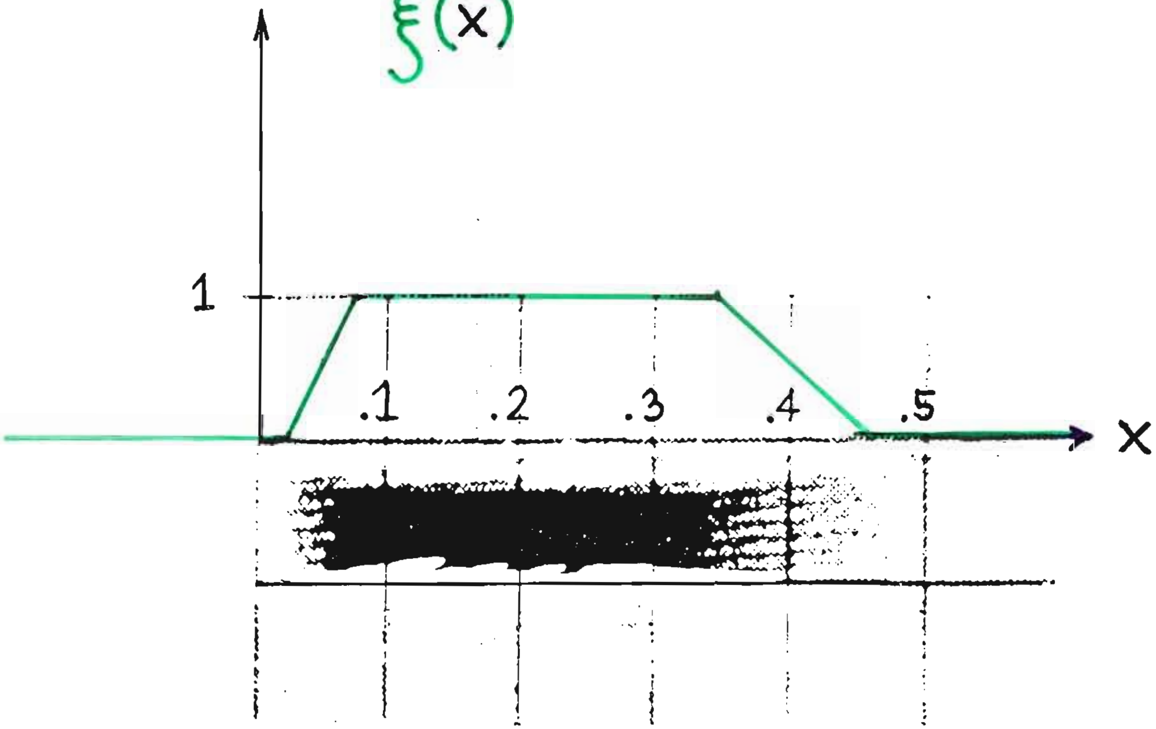
$$C_\delta = \{x \in \mathbb{R} : \xi(x) \geq \delta\} \neq \emptyset$$

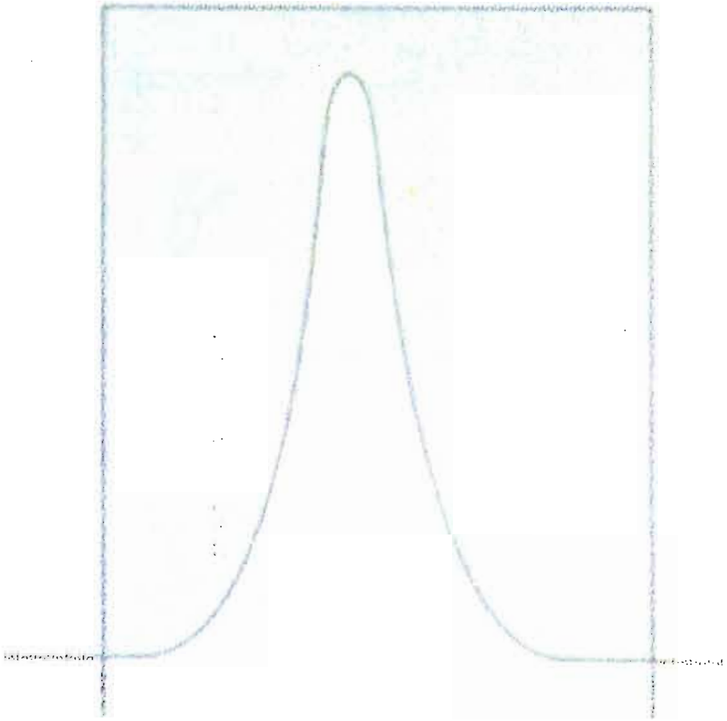
is a closed interval $[a_\delta ; b_\delta]$

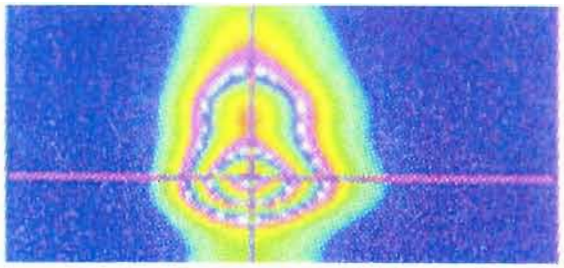

ξ(x)
1
δ
x
a_δ
C_δ
b_δ

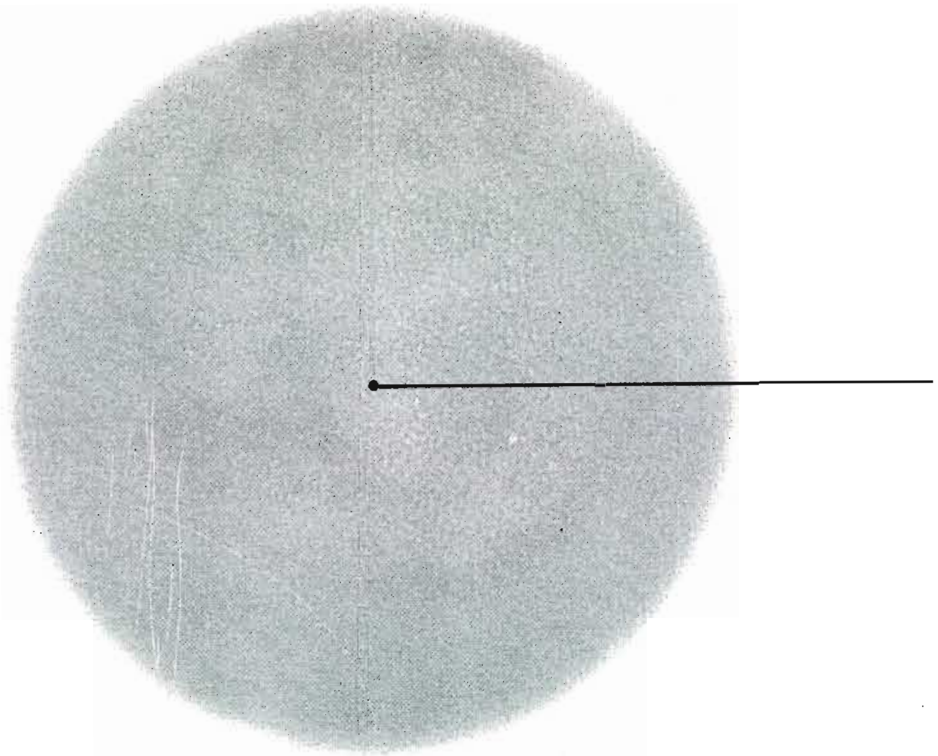
one observation as presented by the screen




ξ(x)
1
.1
.2
.3
.4
.5
x







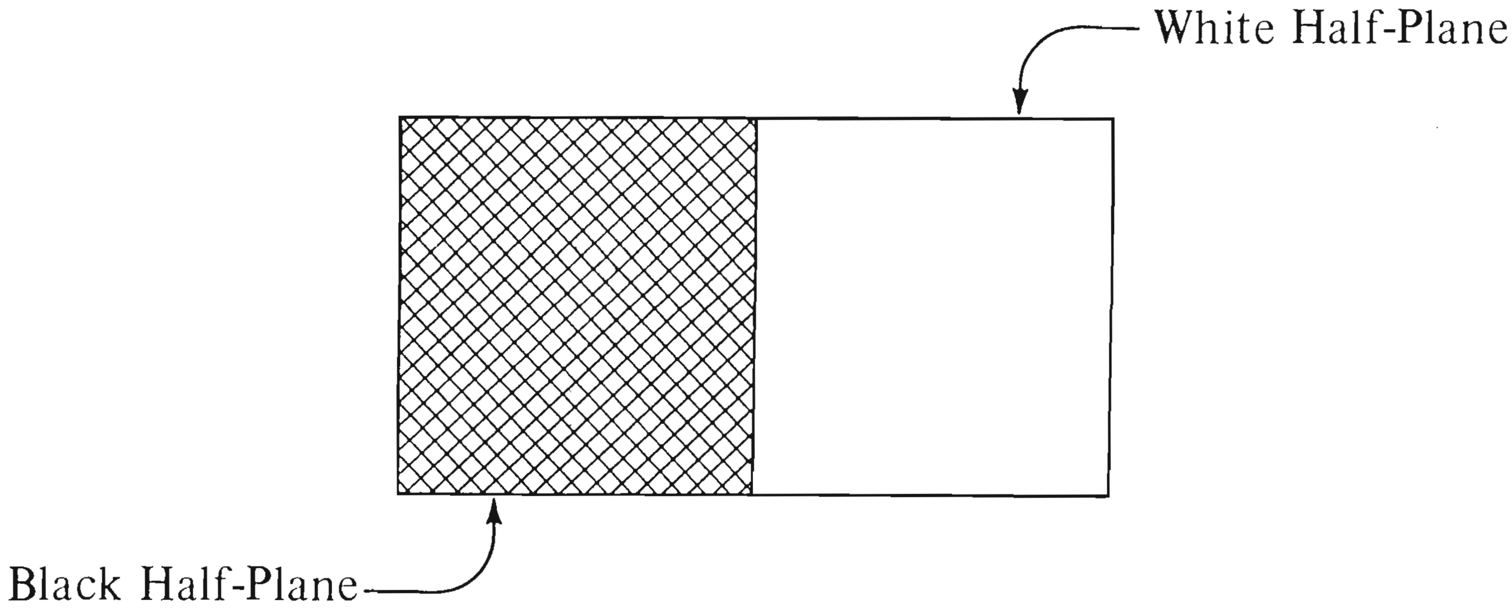


Figure Black and white half-planes.

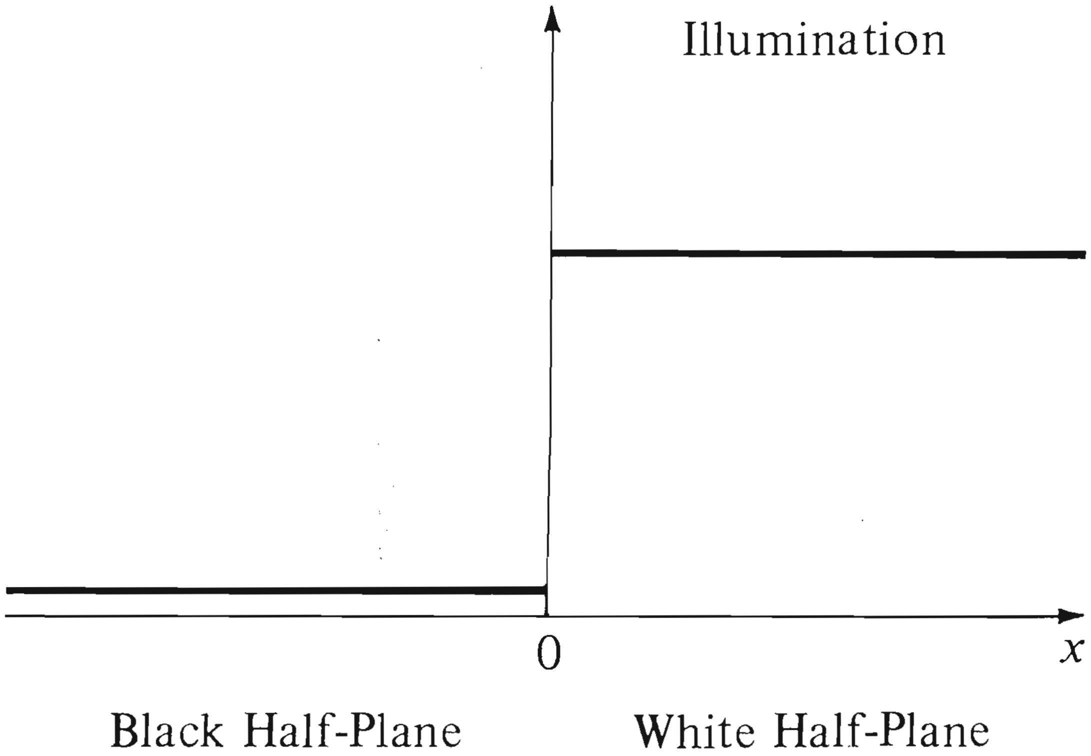


Figure Ideal illumination on a horizontal line

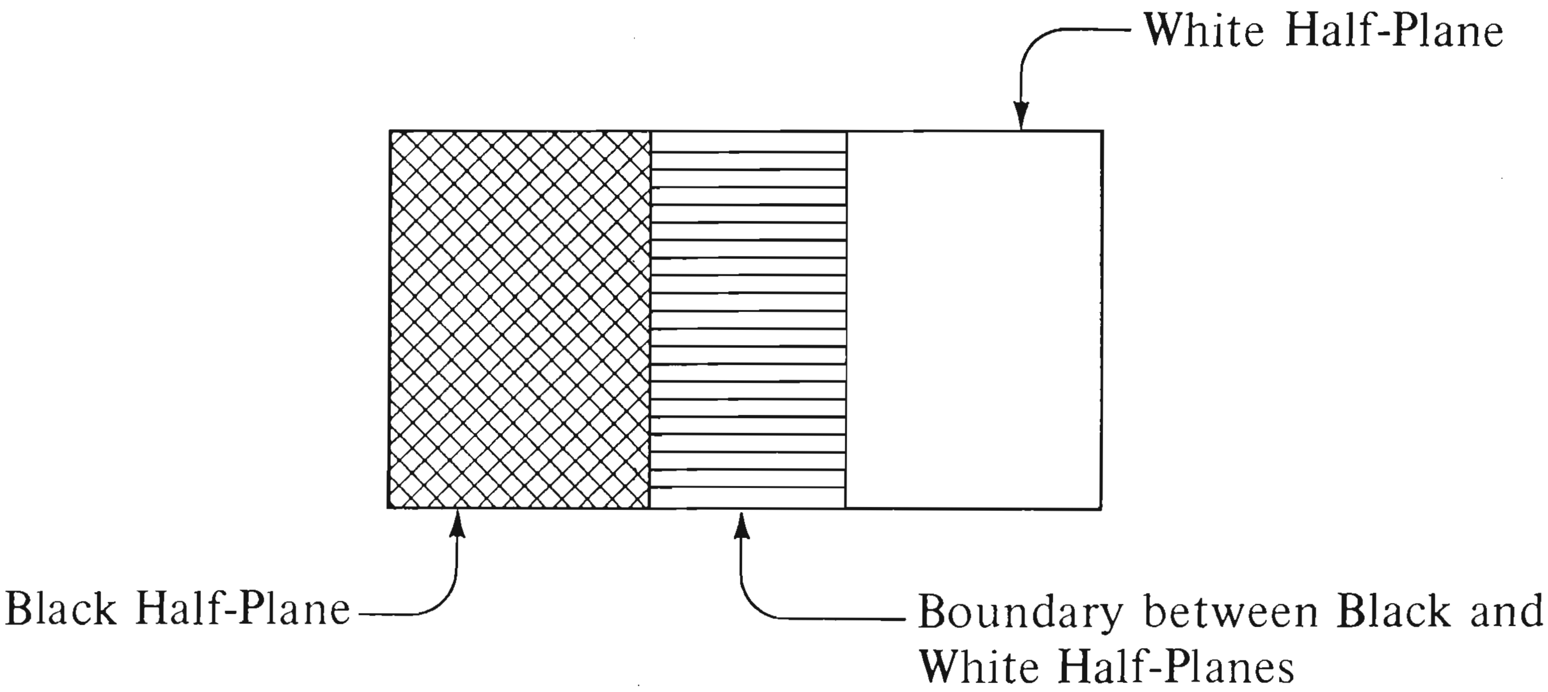
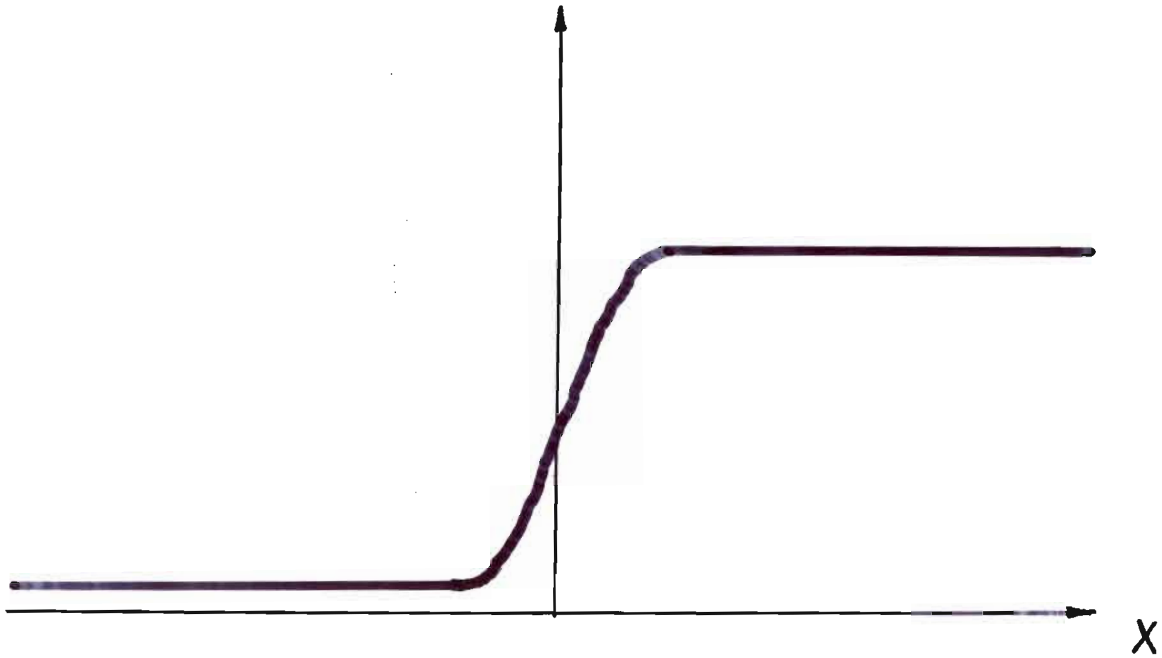
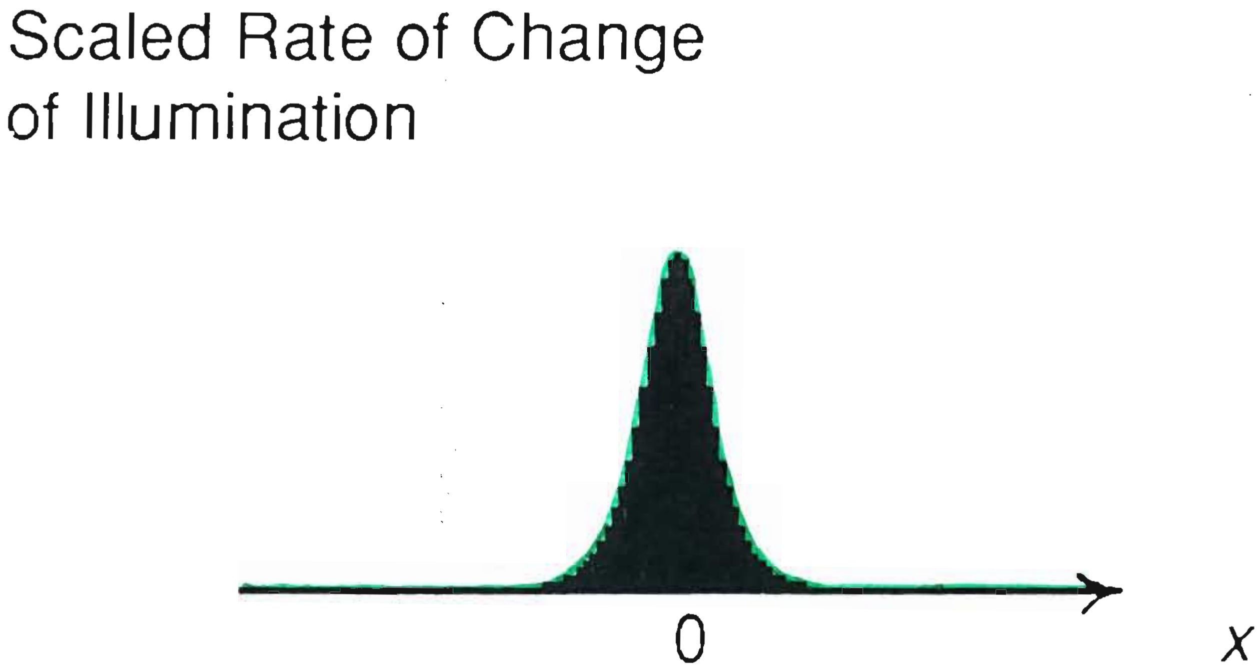


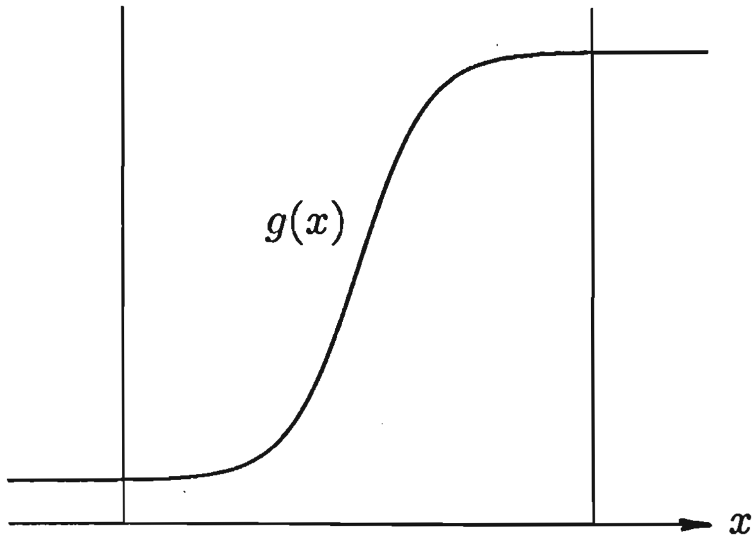
Figure Boundary between half-planes.

Realistic illumination

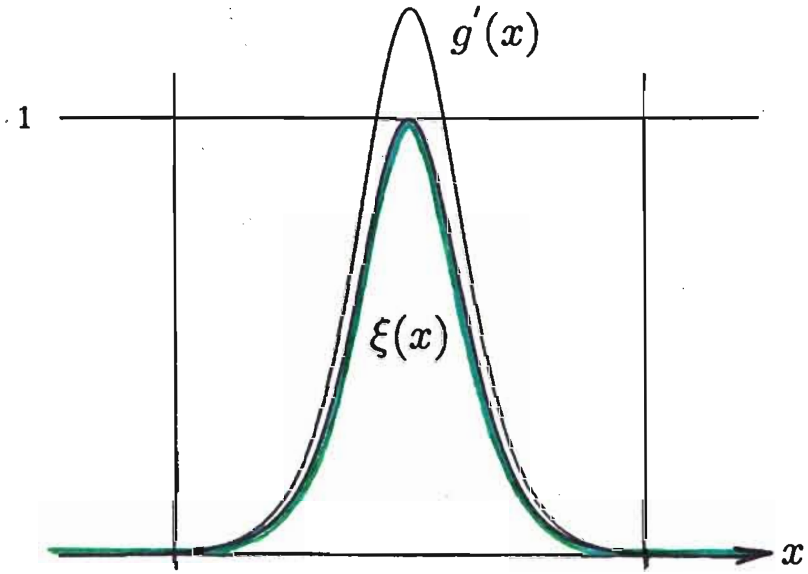




Derivative of illumination function displayed

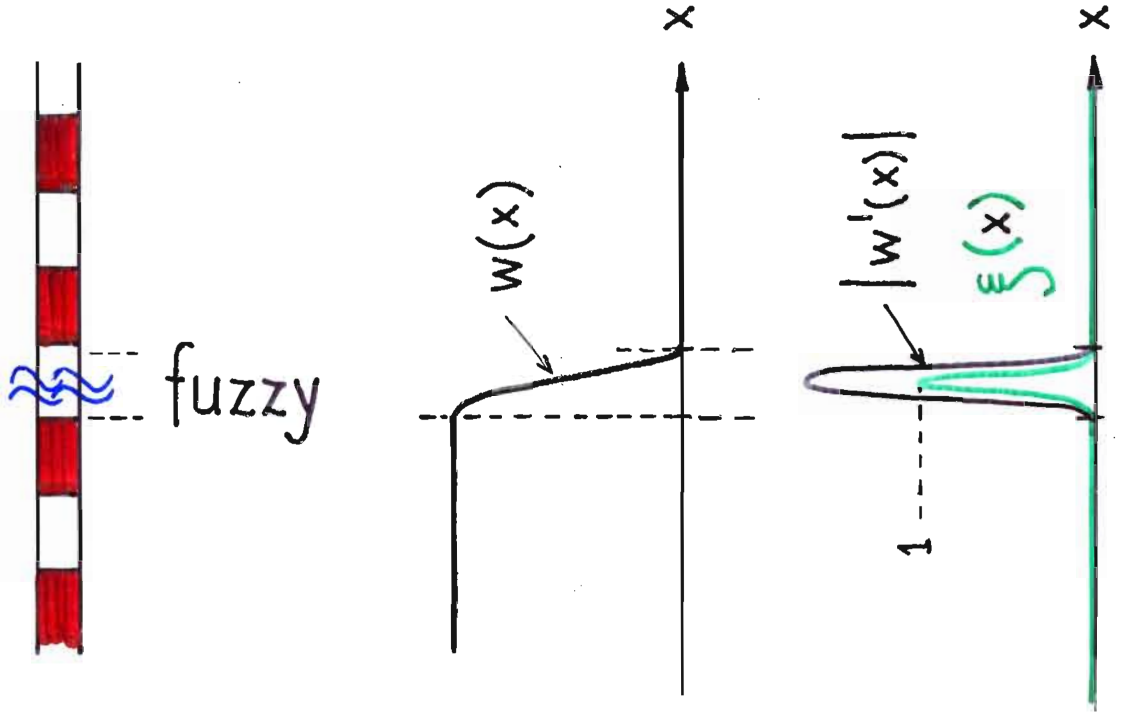


g(x)
x



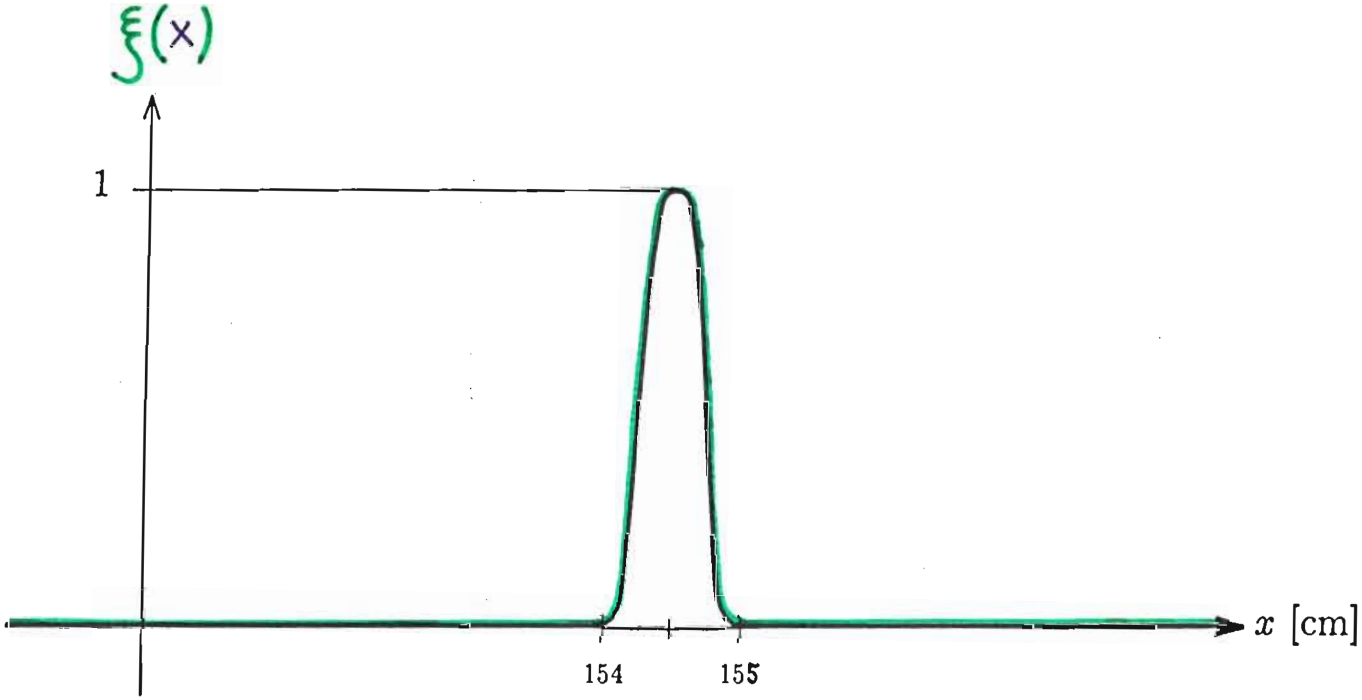
g'(x)
1
ξ(x)
x

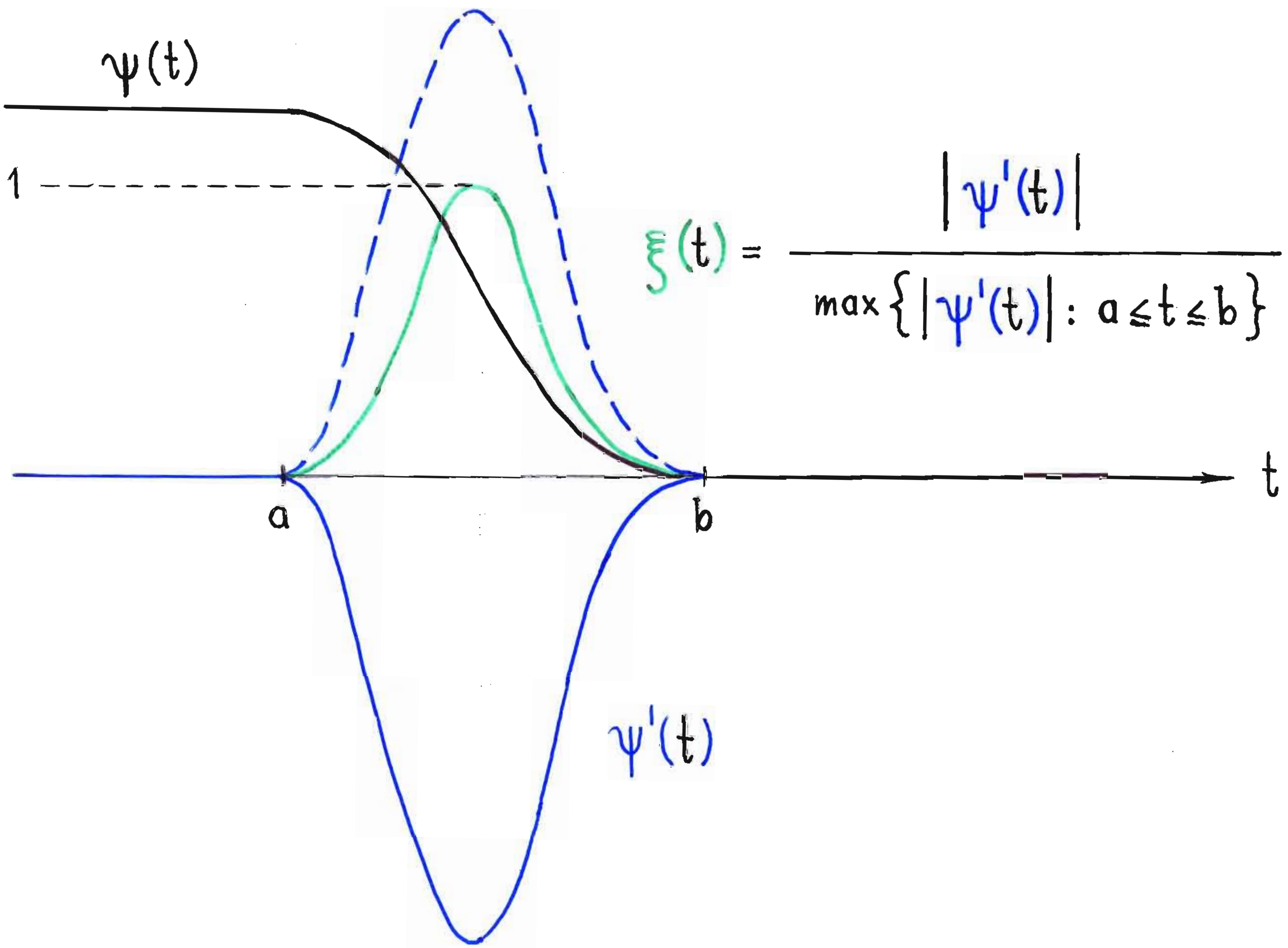
WATER LEVEL



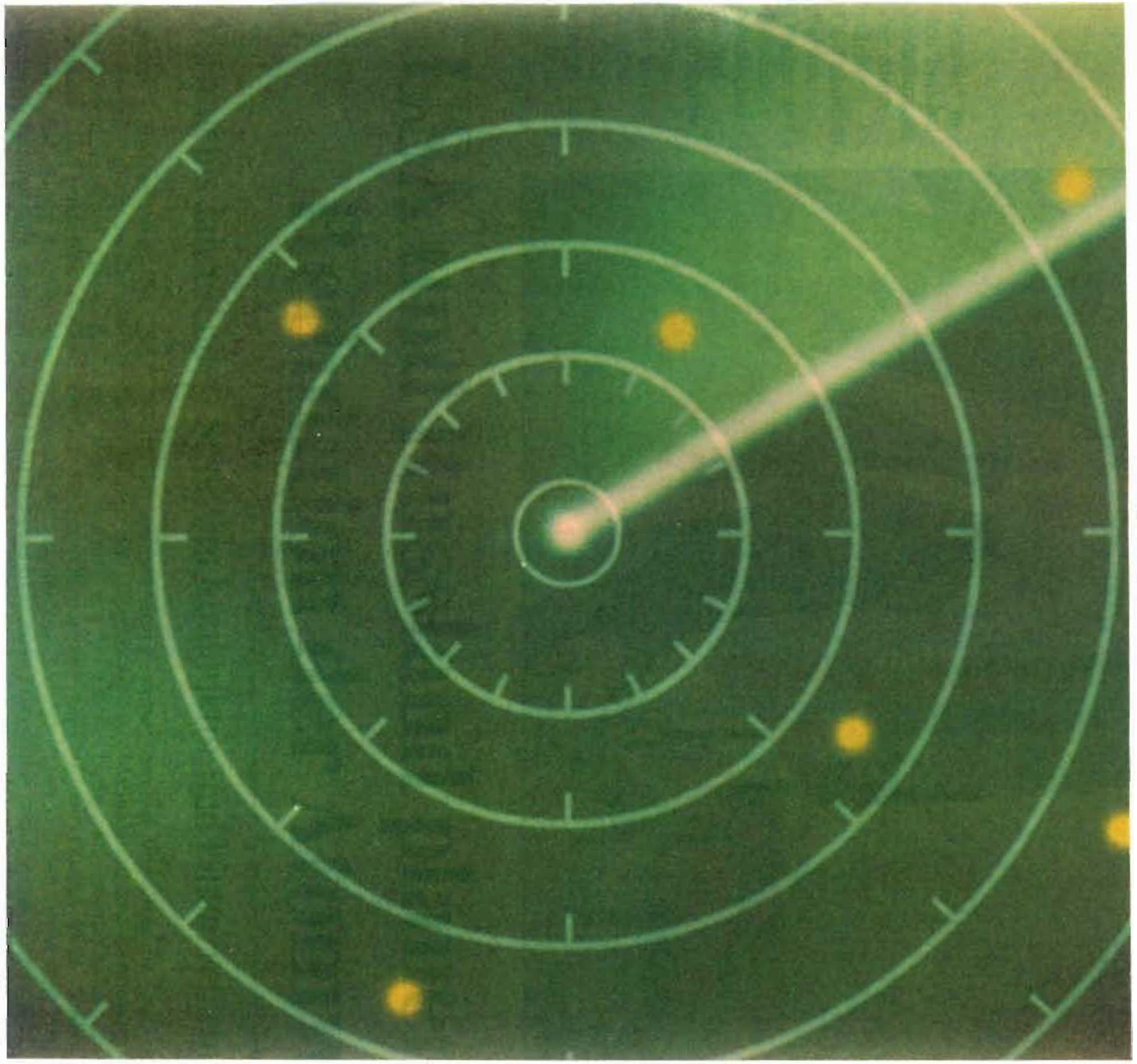
$$\xi(x) = \frac{|w'(x)|}{\max\{|w'(x)| : x \in \mathbb{R}\}} \quad \forall x \in \mathbb{R}$$

Figure : Water level





ψ(t)
1
ξ(t) = |ψ'(t)| / max{|ψ'(t)| : a ≤ t ≤ b}
t
a
b
ψ'(t)



FUZZY VECTOR

vector-characterizing function $\xi(\cdot)$

$\xi : \mathbb{R}^k \longrightarrow [0;1]$

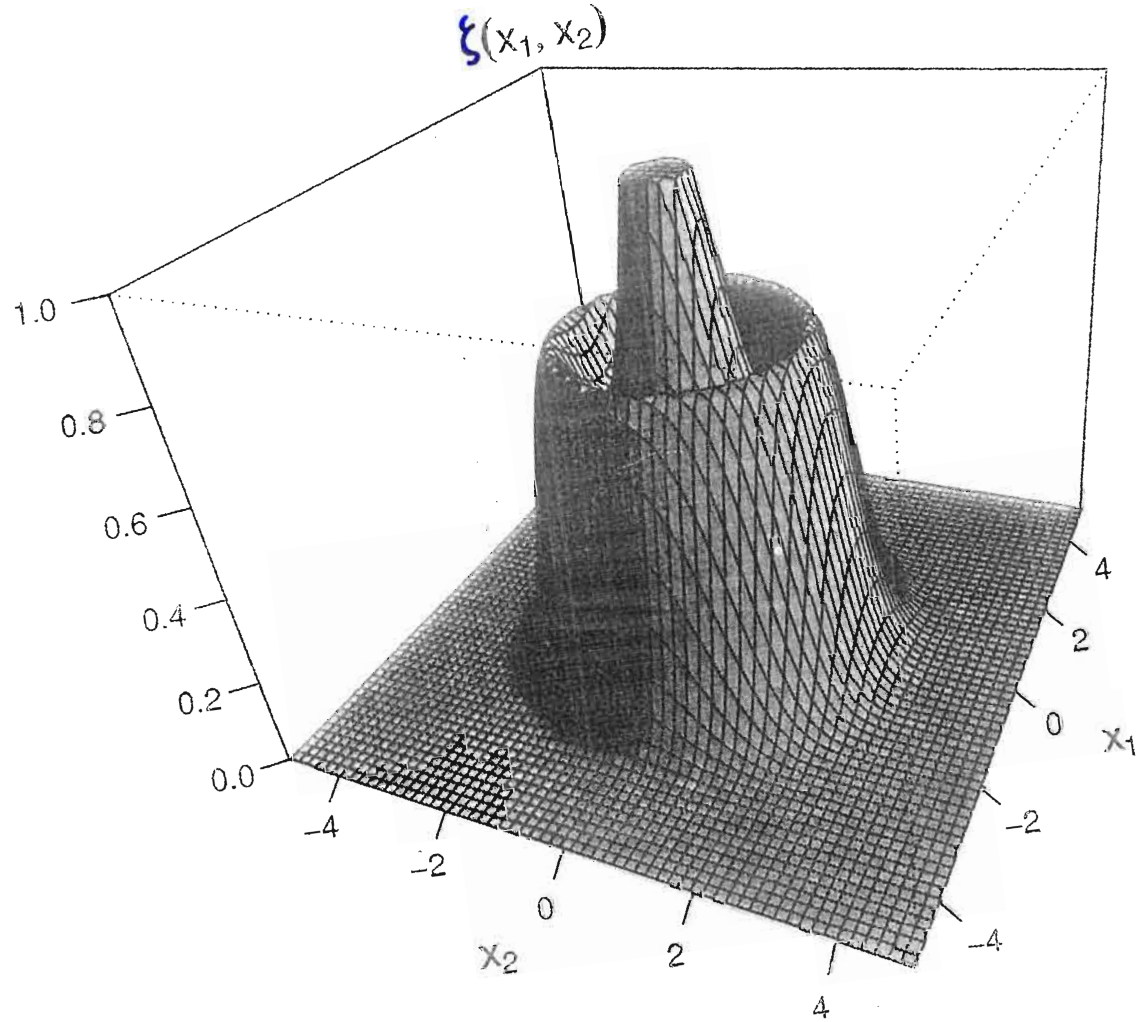
obeying

- $\forall\, \delta \in (0;1]$ the so-called δ-cut

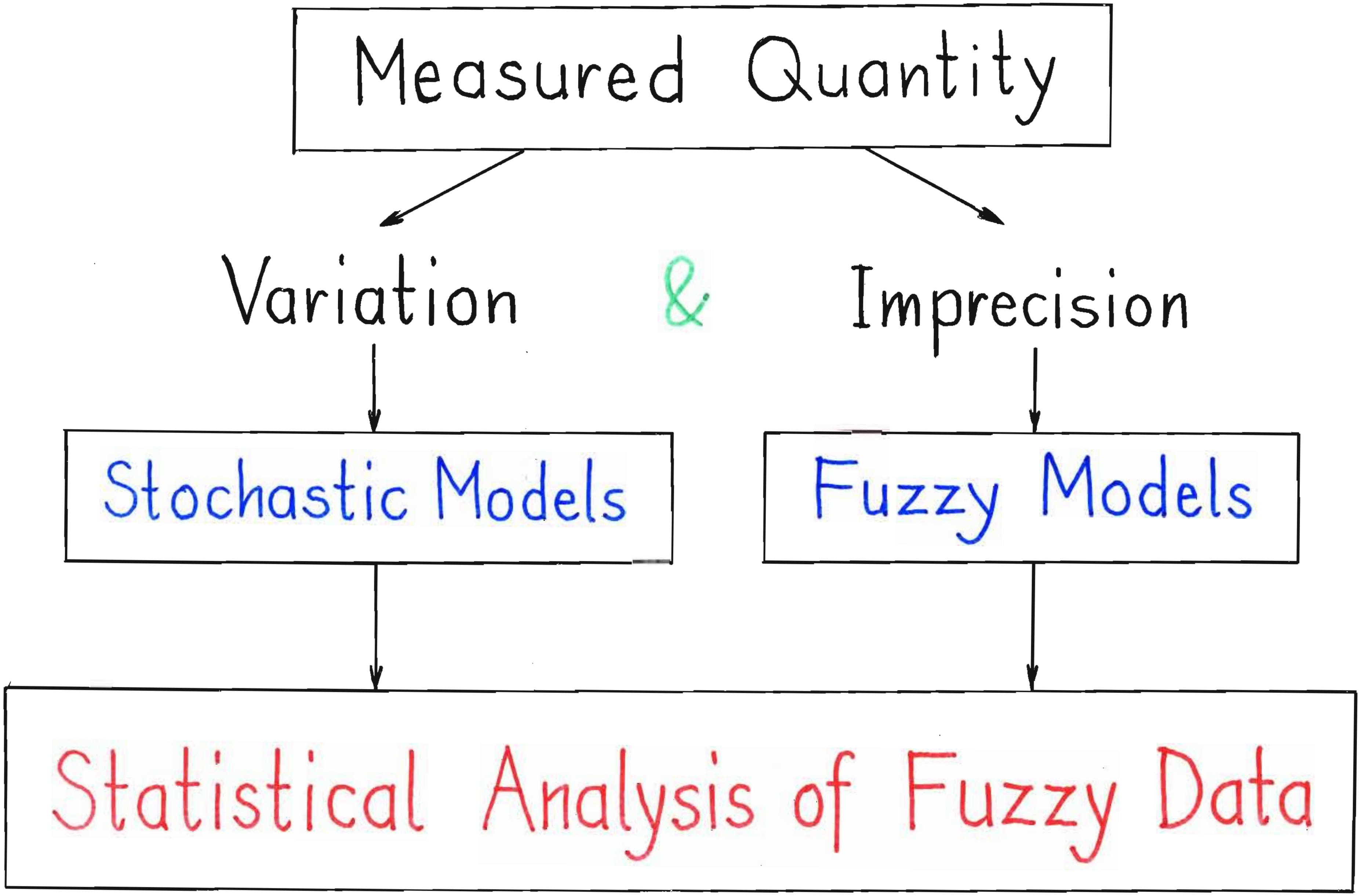
  $$C_\delta[\xi(\cdot)] := \{\underline{x} \in \mathbb{R}^k : \xi(\underline{x}) \geq \delta\} \neq \emptyset$$

  is a finite union of simply connected closed sets

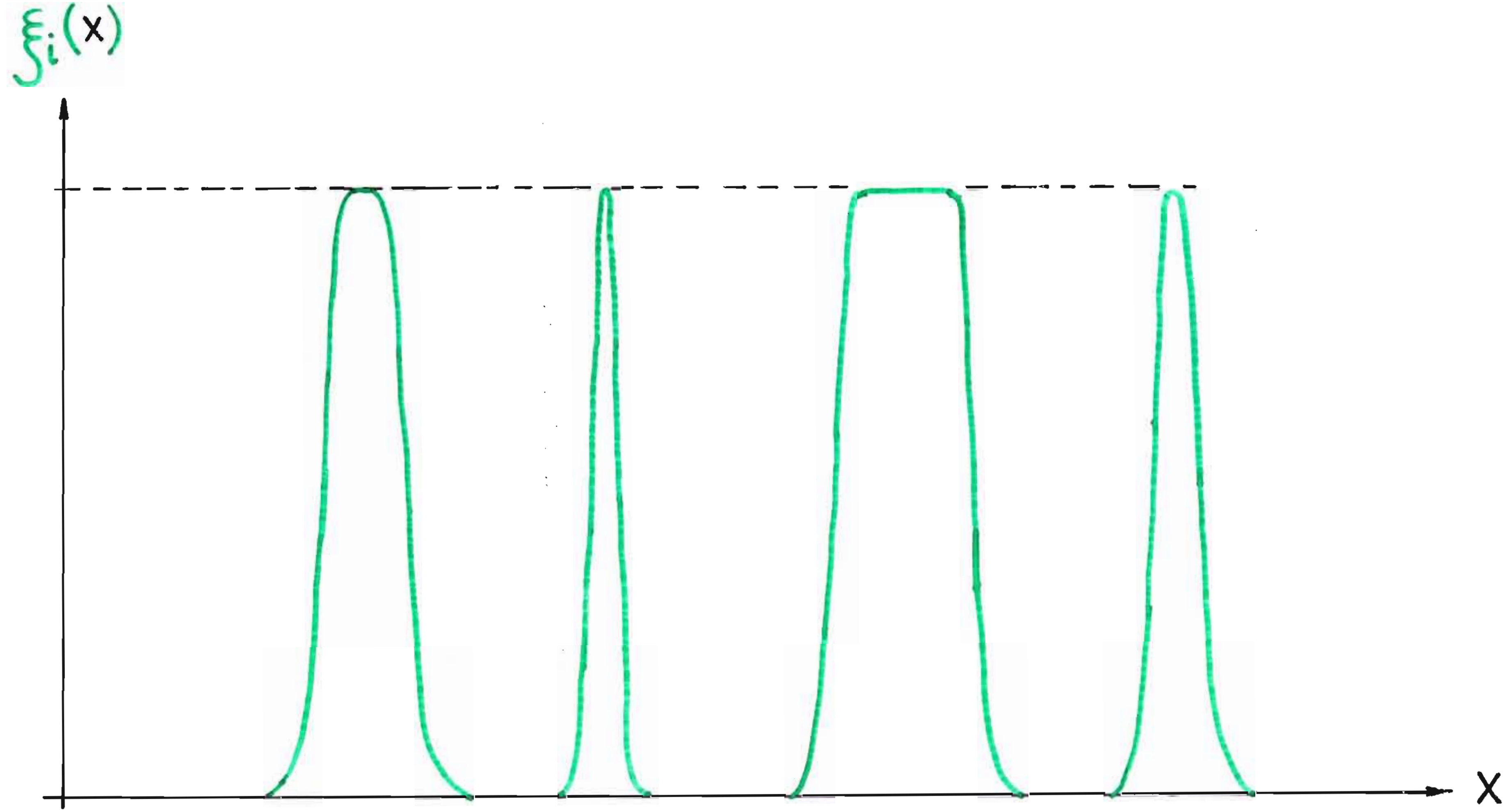
- $\mathrm{Supp}[\xi(\cdot)]$ is bounded



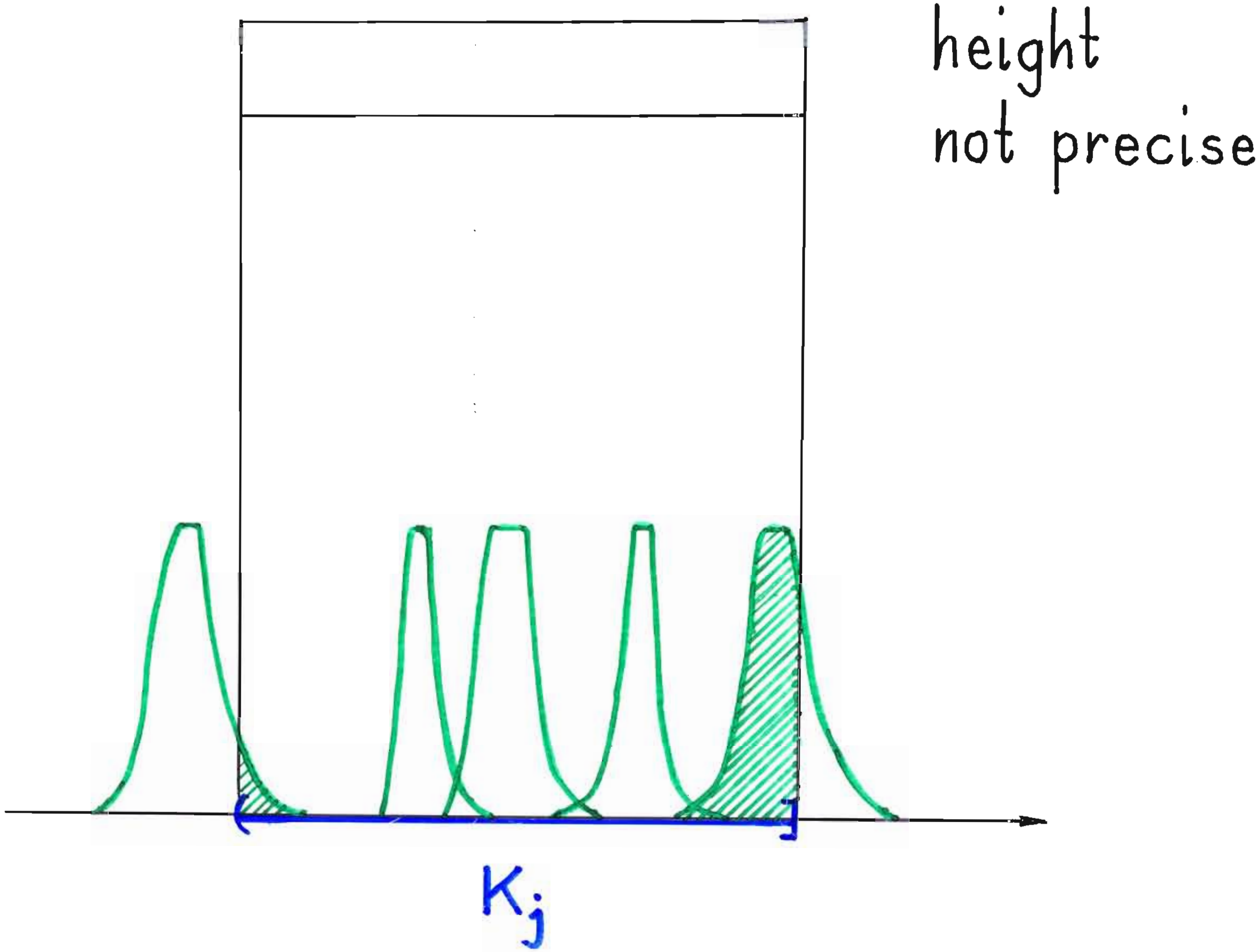
ξ(x1, x2)
1.0
0.8
0.6
0.4
0.2
0.0
-4
-2
0
2
4
x2
4
2
0
-2
-4
x1


Measured Quantity
Variation
&
Imprecision
Stochastic Models
Fuzzy Models
Statistical Analysis of Fuzzy Data

FUZZY SAMPLE



FUZZY HISTOGRAMS



CONSTRUCTION LEMMA

Let $(A_\delta ; \delta \in (0;1])$ be a nested family of subsets of a set M. Then the membership function of the corresponding fuzzy subset of M is given by

$$\xi(x) = \sup\{\delta \cdot \mathbb{1}_{A_\delta}(x) : \delta \in [0;1]\} \quad \forall x \in M$$

with $A_0 := M$

FUZZY FREQUENCY

n_j^* fuzzy absolute frequency of class K_j

δ - Cuts $\quad C_\delta(n_j^*) = [\underline{n}_\delta(K_j), \overline{n}_\delta(K_j)] \quad \forall \delta \in (0;1]$

where

$\overline{n}_\delta(K_j) = \#$ observ. with $C_\delta(\xi_i(\cdot)) \cap K_j \neq \emptyset$

$\underline{n}_\delta(K_j) = \#$ observ. with $C_\delta(\xi_i(\cdot)) \subseteq K_j$

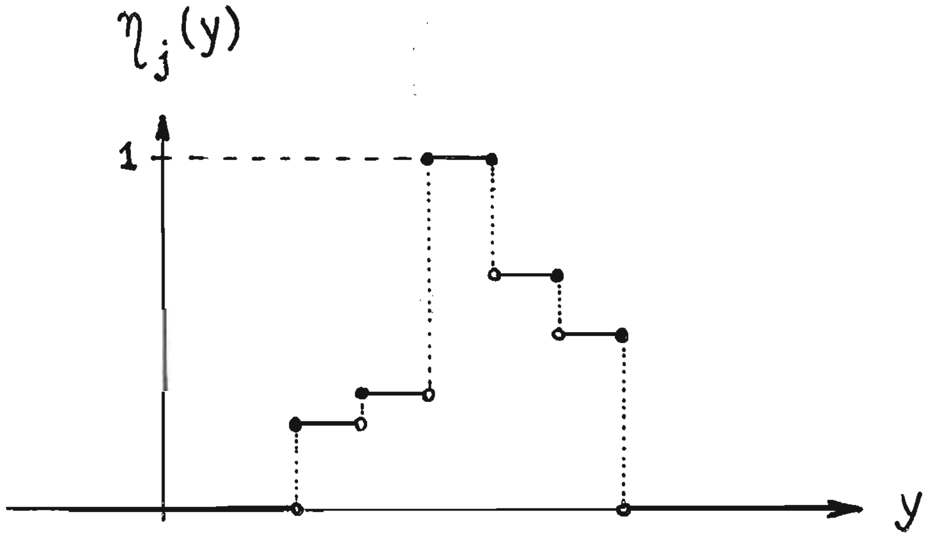
$\Rightarrow$ char. f. $\psi_j(\cdot)$ of n_j^* given by its values

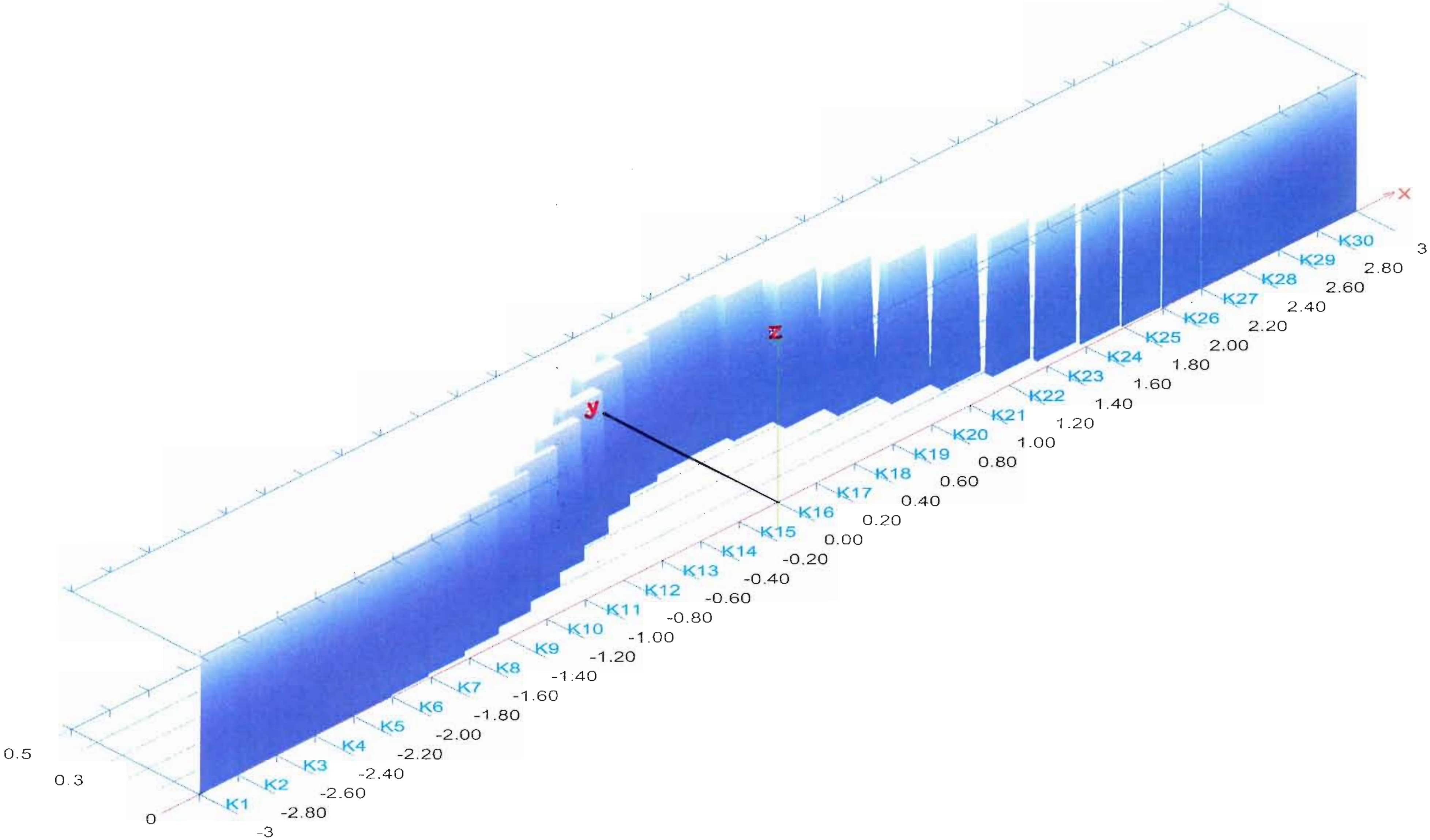
$$\psi_j(y) := \sup_{\delta \in [0,1]} \delta \cdot I_{C_\delta(n_j^*)}(y) \quad \forall y \in \mathbb{R}$$

$h_j^* := \frac{n_j^*}{n}$ fuzzy relative frequency of class K_j

$\Rightarrow$ char. f. $\eta_j(\cdot)$ of h_j^* is given by

$$\eta_j(y) = \psi_j(ny) \quad \forall\, y \in \mathbb{R}$$





x
y
z
K1
K2
K3
K4
K5
K6
K7
K8
K9
K10
K11
K12
K13
K14
K15
K16
K17
K18
K19
K20
K21
K22
K23
K24
K25
K26
K27
K28
K29
K30
-3
-2.80
-2.60
-2.40
-2.20
-2.00
-1.80
-1.60
-1.40
-1.20
-1.00
-0.80
-0.60
-0.40
-0.20
0.00
0.20
0.40
0.60
0.80
1.00
1.20
1.40
1.60
1.80
2.00
2.20
2.40
2.60
2.80
3
0
0.3
0.5

CALCULATIONS

Sums $\sum_{i=1}^{n} x_i^*$

Averages $\bar{x}_n^*$

Indicators and Indexes I^*

$$I^* = f(x_1^*, \dots, x_n^*; w_1, \dots, w_n)$$

Functions of Fuzzy Variables

Extension Principle

EXTENSION PRINCIPLE

$g: M \longrightarrow N, \quad x \in M \Rightarrow g(x) \in N$

for fuzzy $x^* \triangleq \xi(\cdot) \Rightarrow g(x^*)$ fuzzy

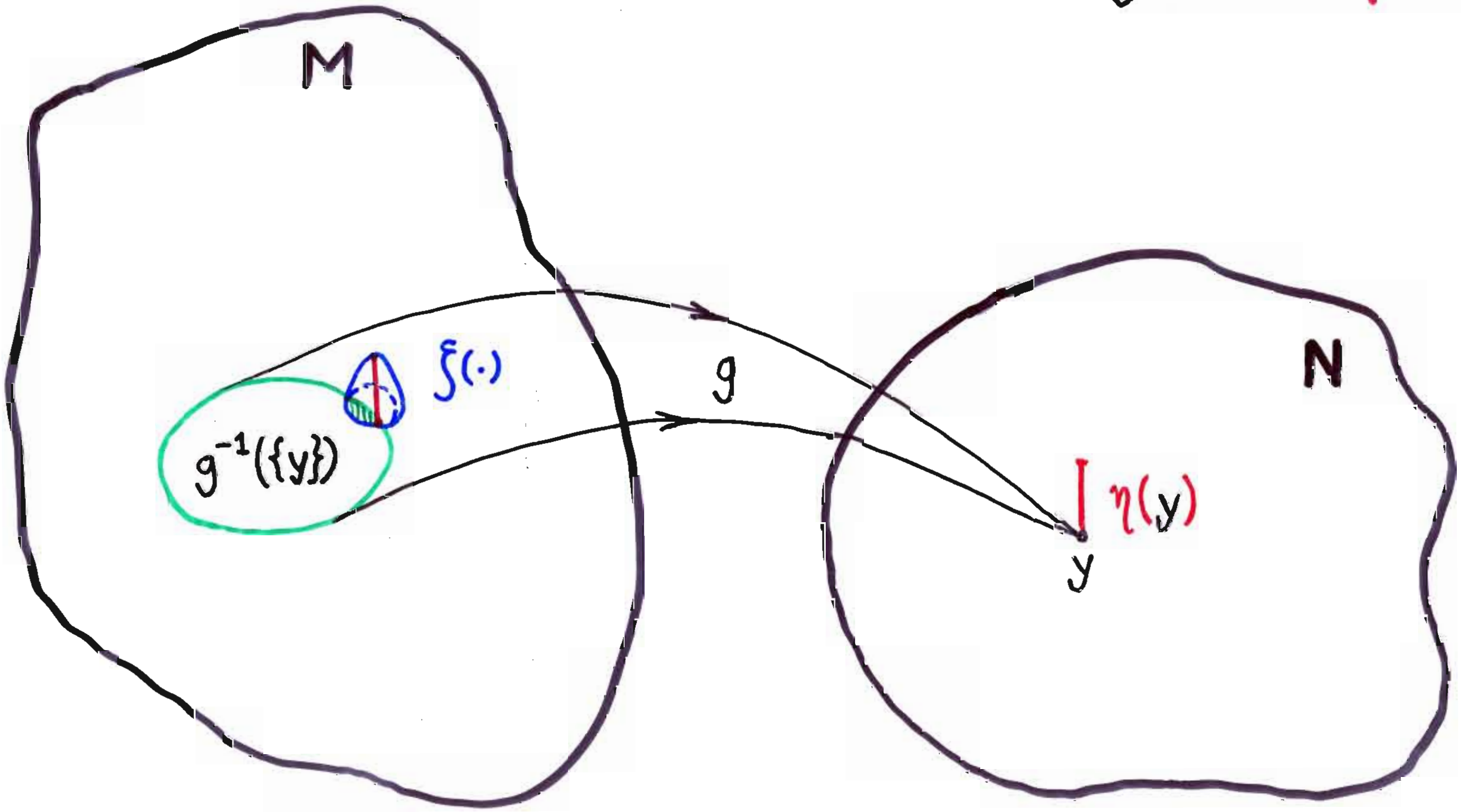
$\eta(\cdot)$ membership function of $y^* = g(x^*)$

$$\eta(y) = \begin{cases} \sup\{\xi(x) : g(x) = y\} & \text{if } g^{-1}(\{y\}) \neq \emptyset \\ 0 & \text{if } g^{-1}(\{y\}) = \emptyset \end{cases} \quad \forall y \in N$$

Extension: $g: \mathcal{F}(M) \longrightarrow \mathcal{F}(N)$

$$g: M \longrightarrow N, \quad y = g(x) \quad \forall x \in M$$

$$x^* \triangleq \xi(\cdot), \quad g(x^*) \triangleq \eta(\cdot)$$



$$g^{-1}(\{y\}) := \{x \in M : g(x) = y\}$$

STANDARD STATISTICAL INFERENCE

$X \sim P_\theta ; \ \theta \in \Theta$, M_X Observation Space

$x_1, \dots, x_n$ Sample , $x_i \in M_X \Rightarrow (x_1, \dots, x_n) \in M_X^n$

M_X^n Sample Space

- Estimators $\vartheta(x_1, \dots, x_n)$, $\vartheta : M_X^n \to \Theta$
- Confidence Regions $\kappa(x_1, \dots, x_n)$
- Test Statistics $t(x_1, \dots, x_n)$

Generalization for Fuzzy Data ?

COMBINED FUZZY SAMPLE

Sample $x_1^*, \dots, x_n^*$

$\xi_1(\cdot), \dots, \xi_n(\cdot)$

x_i^* Fuzzy Element of Observation Space M

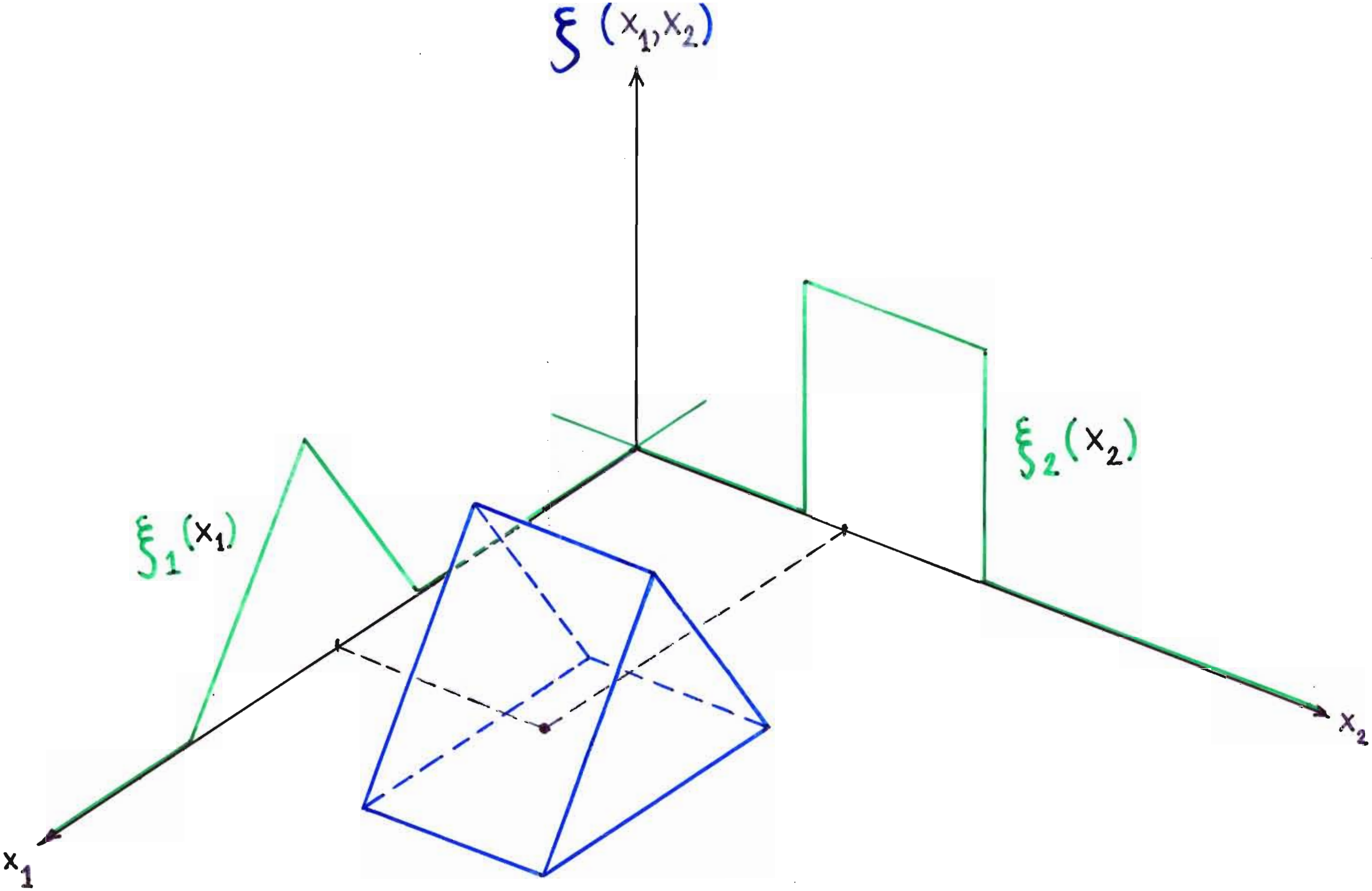
$M^n = \{ \underline{x} = (x_1, \dots, x_n) : x_i \in M \}$ Sample Space

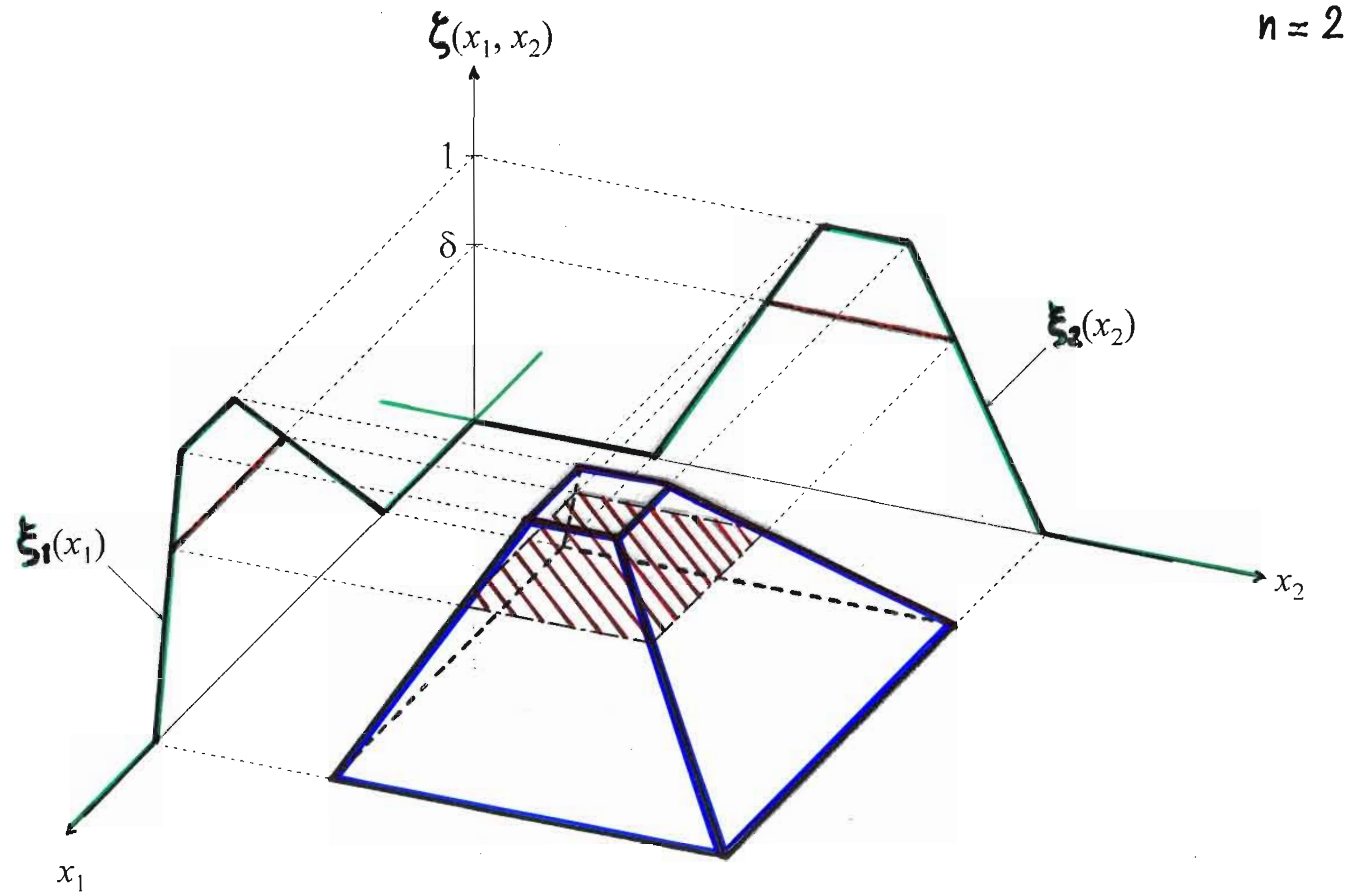
$\underline{x}^*$ Fuzzy Element of M^n with VCF $\xi(\cdot)$

$$\xi(x_1, \dots, x_n) = T_n\left(\xi_1(x_1), \dots, \xi_n(x_n)\right) \quad \forall (x_1, \dots, x_n)$$

$\underline{x}^*$ Combined Fuzzy Sample

$n = 2$





$$\xi(\underline{x}) := \min_{i=1(1)n} \xi_i(x_i)$$

BAYESIAN INFERENCE

$X \sim f(\cdot|\theta)$, $\theta \in \Theta$, $\tilde{\theta}$ Stochastic Qu.

$\pi(\cdot)$ a-priori distribution on Θ

$x_1, \cdots, x_n$ Sample information

Updating of the a-priori distribution

$$\pi(\theta|x_1,\cdots,x_n) = \frac{\pi(\theta)\cdot \ell(\theta; x_1,\cdots,x_n)}{\int_\Theta \pi(\theta)\cdot \ell(\theta; x_1,\cdots,x_n)\,d\theta} \quad \forall\, \theta \in \Theta$$

a-posteriori distribution

$$\ell(\theta; x_1,\cdots,x_n) = \prod_{i=1}^{n} f(x_i|\theta)$$

FOR FUZZY DATA ?

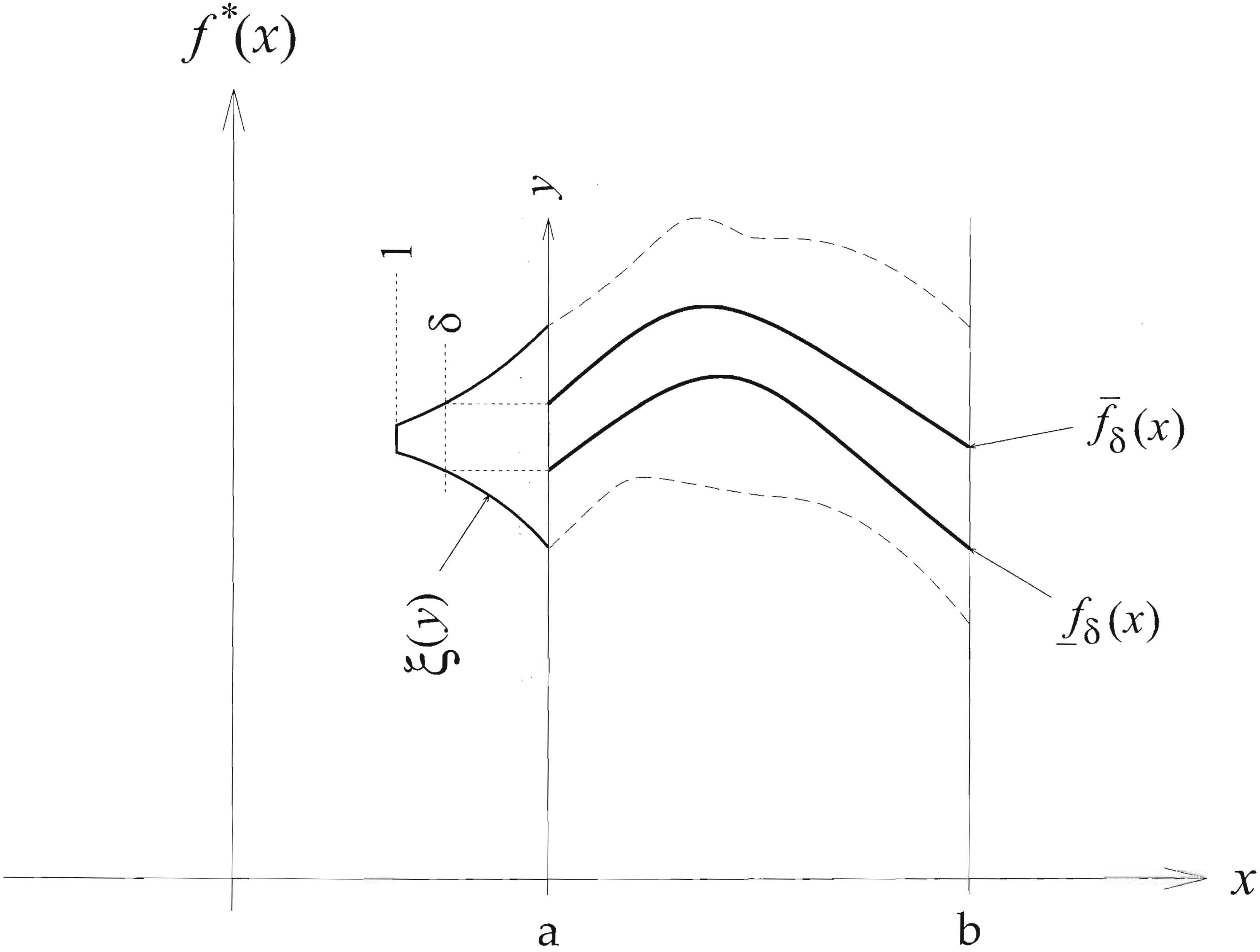
Fuzzy valued functions $f^*: M \to \mathcal{F}_I(\mathbb{R})$

$$f^*(x) = y^* \mathrel{\hat{=}} \xi_x(\cdot) \quad \forall x \in M$$

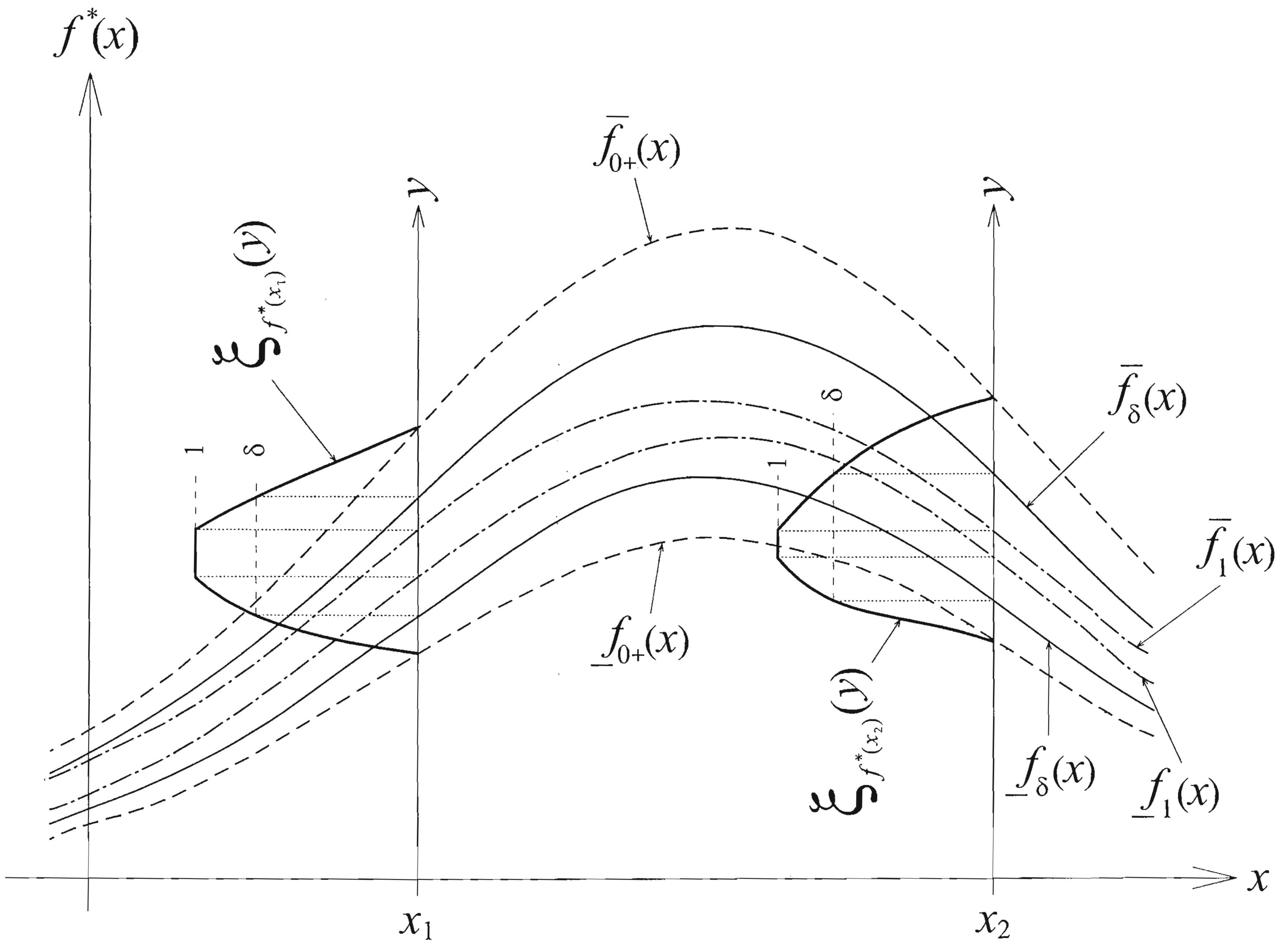
δ- level functions $\underline{f}_\delta(\cdot)$ and $\overline{f}_\delta(\cdot)$

defined by $C_\delta[f^*(x)] = [\underline{f}_\delta(x), \overline{f}_\delta(x)] \quad \forall x \in M$
$\forall \delta \in (0;1]$

For $M = \mathbb{R}$ δ-level curves (real functions)



f*(x)
y
1
δ
ξ(y)
f̄δ(x)
f̲δ(x)
x
a
b



$f^*(x)$
$\bar{f}_{0+}(x)$
y
y
$\xi_{f^*(x_1)}(y)$
1
δ
δ
1
$\bar{f}_\delta(x)$
$\bar{f}_1(x)$
$\underline{f}_{0+}(x)$
$\xi_{f^*(x_2)}(y)$
$\underline{f}_\delta(x)$
$\underline{f}_1(x)$
x
x_1
x_2

PROBLEMS

Sequential updating

Precise a-priori density

ALTERNATIVE SOLUTION

Based on δ-level functions

$$\overline{\pi}_\delta(\cdot), \quad \overline{\ell}_\delta(\cdot\,;\underline{x}^*), \quad \overline{\pi}_\delta(\cdot\,|\,\underline{x}^*)$$

$$\underline{\pi}_\delta(\cdot), \quad \underline{\ell}_\delta(\cdot\,;\underline{x}^*), \quad \underline{\pi}_\delta(\cdot\,|\,\underline{x}^*)$$

FUZZY PROBABILITY DENSITY

Generalized densities $f^*(\cdot)$ on $\mathbb{R}$:

$f^*(\cdot)$ fuzzy function with δ-level functions $\underline{f}_\delta(\cdot)$ and $\overline{f}_\delta(\cdot)$ integrable with

$$\int_{\mathbb{R}} \overline{f}_\delta(x)\,dx < \infty \quad \forall \delta \in (0;1]$$

and $\exists$ classical density $f(\cdot)$ on $\mathbb{R}$ with

$$\underline{f}_1(x) \leq f(x) \leq \overline{f}_1(x) \quad \forall x \in \mathbb{R}$$

The fuzzy probability $P^*(B)$ of $B \in \mathcal{B}$ is a fuzzy interval

LIKELIHOOD FOR FUZZY DATA

$\underline{x}^*$ combined fuzzy sample with v.c.f. $\xi(\cdot)$

$\ell^*(\theta;\underline{x}^*)$ fuzzy value of the likelihood $\ell(\theta;\underline{x})$ with c.f. $\eta_\theta(\cdot)$ defined by

$$\eta_\theta(y) = \begin{cases} \sup\{\xi(\underline{x}) : \ell(\theta;\underline{x}) = y\} & \text{if } \ell^{-1}(\{y\}) \neq \emptyset \\ 0 & \text{if } \ell^{-1}(\{y\}) = \emptyset \end{cases} \quad \forall y \in \mathbb{R}$$

Remark: For precise data $\underline{x}$ the indicator function of $\ell(\theta;\underline{x})$ is obtained

GENERALIZED BAYES' THEOREM

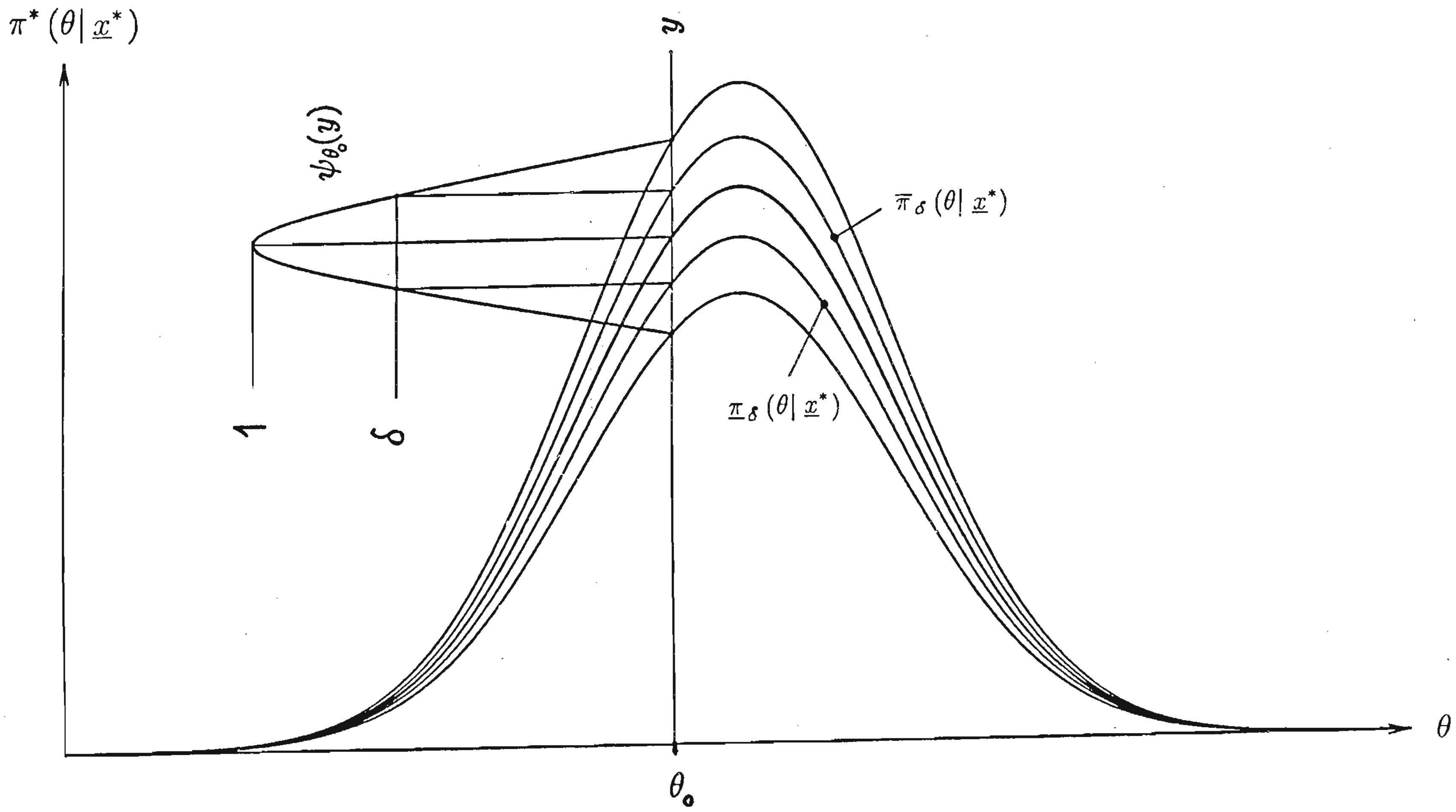
δ - level curves of the fuzzy a-posteriori density

$$\overline{\pi}_\delta(\theta \mid \underline{x}^*) = \frac{\overline{\pi}_\delta(\theta)\,\overline{\ell}_\delta(\theta;\underline{x}^*)}{\int_\Theta \frac{1}{2}\left[\underline{\pi}_\delta(\theta)\,\underline{\ell}_\delta(\theta;\underline{x}^*) + \overline{\pi}_\delta(\theta)\,\overline{\ell}_\delta(\theta;\underline{x}^*)\right] d\theta}$$

$$\underline{\pi}_\delta(\theta \mid \underline{x}^*) = \frac{\underline{\pi}_\delta(\theta)\,\underline{\ell}_\delta(\theta;\underline{x}^*)}{\int_\Theta \frac{1}{2}\left[\underline{\pi}_\delta(\theta)\,\underline{\ell}_\delta(\theta;\underline{x}^*) + \overline{\pi}_\delta(\theta)\,\overline{\ell}_\delta(\theta;\underline{x}^*)\right] d\theta}$$

$\forall\, \theta \in \Theta$

Figure *Fuzzy a-posteriori density*



EXAMPLE $X \sim Ex_\theta \,, \quad \theta \in \Theta = (0, \infty)$

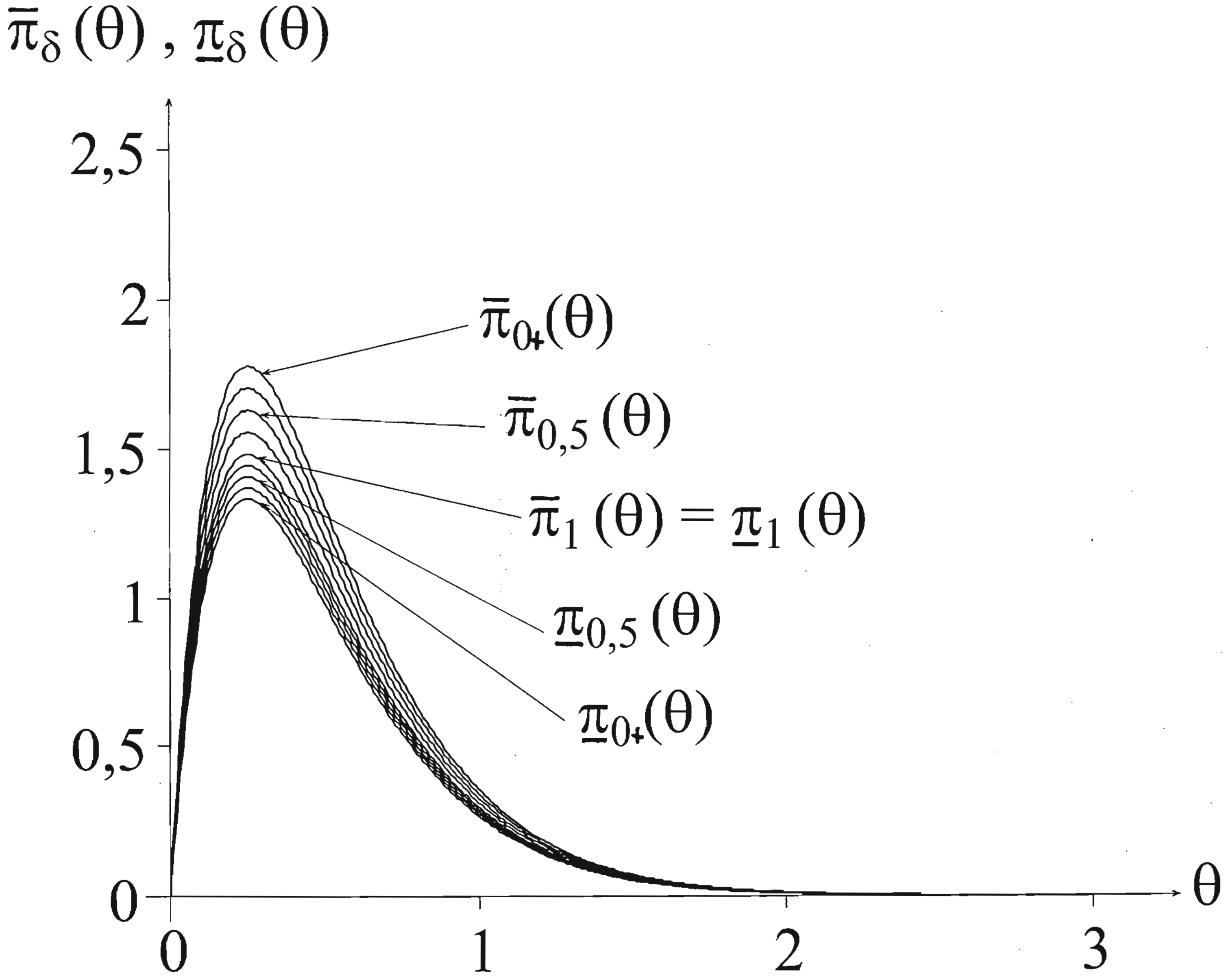
$$f(x|\theta) = \frac{1}{\theta} \exp\left\{-\frac{x}{\theta}\right\} \cdot I_{(0,\infty)}(x)$$

Fuzzy a-priori distribution

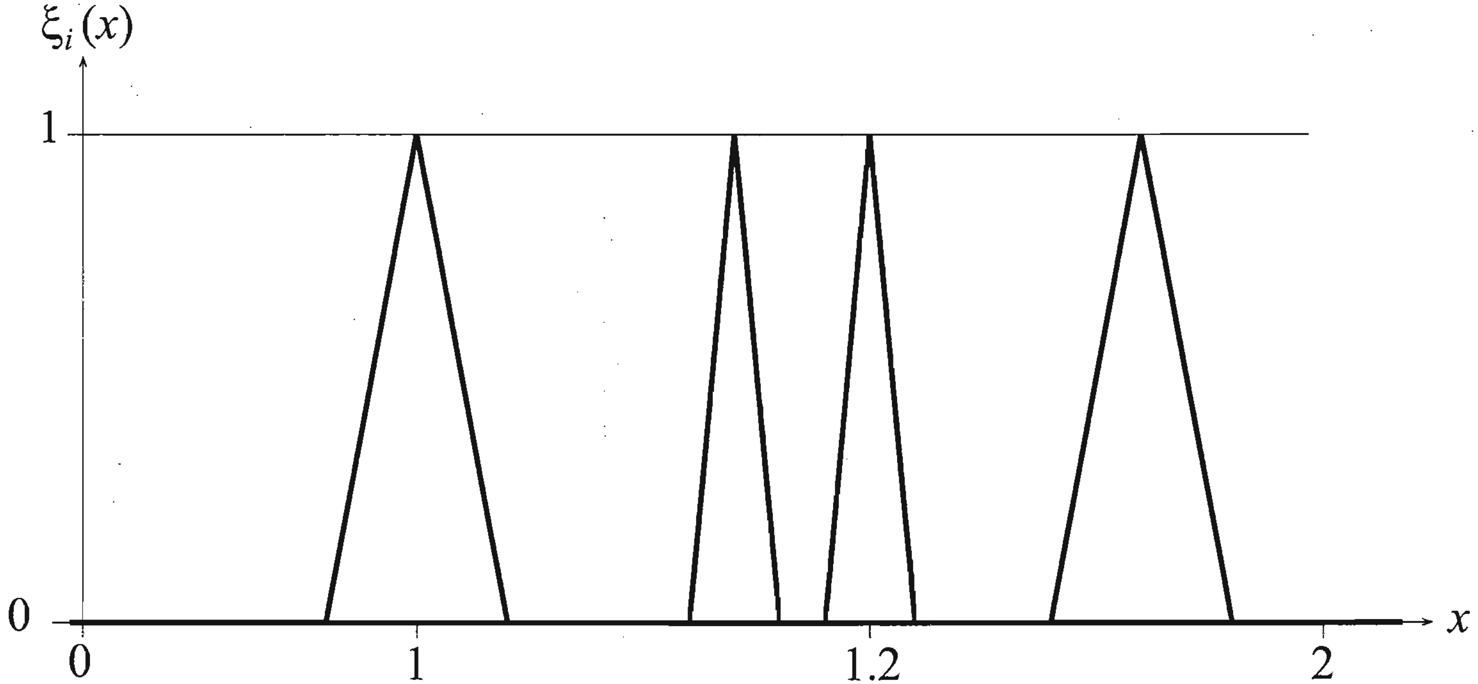
$\pi^*(\cdot)$ fuzzy gamma density

$\overline{\pi}_\delta(\cdot)$ upper
$\underline{\pi}_\delta(\cdot)$ lower

} δ-level curves



$\overline{\pi}_\delta(\theta)$, $\underline{\pi}_\delta(\theta)$
2,5
2
1,5
1
0,5
0
$\overline{\pi}_{0+}(\theta)$
$\overline{\pi}_{0,5}(\theta)$
$\overline{\pi}_1(\theta) = \underline{\pi}_1(\theta)$
$\underline{\pi}_{0,5}(\theta)$
$\underline{\pi}_{0+}(\theta)$
θ
0
1
2
3



ξ_i(x)
1
0
x
0
1
1.2
2

COMBINED FUZZY SAMPLE

$$\underline{x}^* = (x_1, x_2, x_3, x_4)^*$$

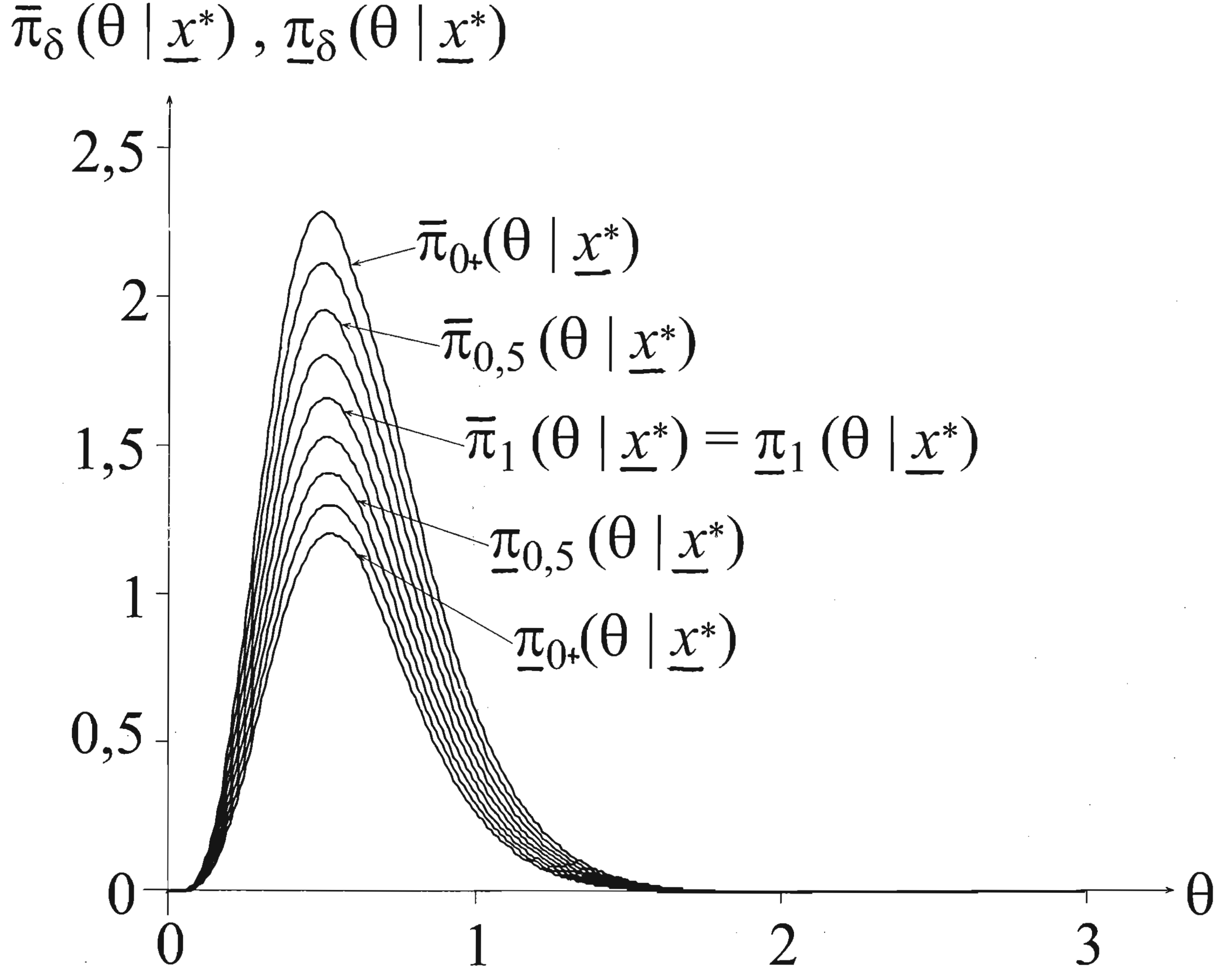
vector char. function $\xi(\cdot,\cdot,\cdot,\cdot)$

$$\xi(x_1, x_2, x_3, x_4) = \min\{\xi_1(x_1), \xi_2(x_2), \xi_3(x_3), \xi_4(x_4)\}$$

$\overline{\pi}_\delta(\cdot \mid \underline{x}^*)$

$\underline{\pi}_\delta(\cdot \mid \underline{x}^*)$

by gen. Bayes' theorem



$\overline{\pi}_\delta(\theta \mid \underline{x}^*)$, $\underline{\pi}_\delta(\theta \mid \underline{x}^*)$
2,5
2
1,5
1
0,5
0
0
1
2
3
θ
$\overline{\pi}_{0+}(\theta \mid \underline{x}^*)$
$\overline{\pi}_{0,5}(\theta \mid \underline{x}^*)$
$\overline{\pi}_1(\theta \mid \underline{x}^*) = \underline{\pi}_1(\theta \mid \underline{x}^*)$
$\underline{\pi}_{0,5}(\theta \mid \underline{x}^*)$
$\underline{\pi}_{0+}(\theta \mid \underline{x}^*)$

HPD - Regions

$\pi(\cdot|D)$ a-posteriori Density

$1-\alpha$ Confidence level

$\Theta_{1-\alpha} \subseteq \Theta$ obeying:

1) $\int_{\Theta_{1-\alpha}} \pi(\theta|D)\, d\theta = 1-\alpha$

2) $\pi(\theta|D)$ max. on $\Theta_{1-\alpha}$

GENERALIZED HPD-Regions

$\pi^*(\cdot | D^*)$ Fuzzy a-posteriori Density

$\mathcal{D}_\delta := \{ g : g \text{ density with } \underline{\pi}_\delta(\theta) \leq g(\theta) \leq \overline{\pi}_\delta(\theta) \;\; \forall \theta \in \Theta \}$

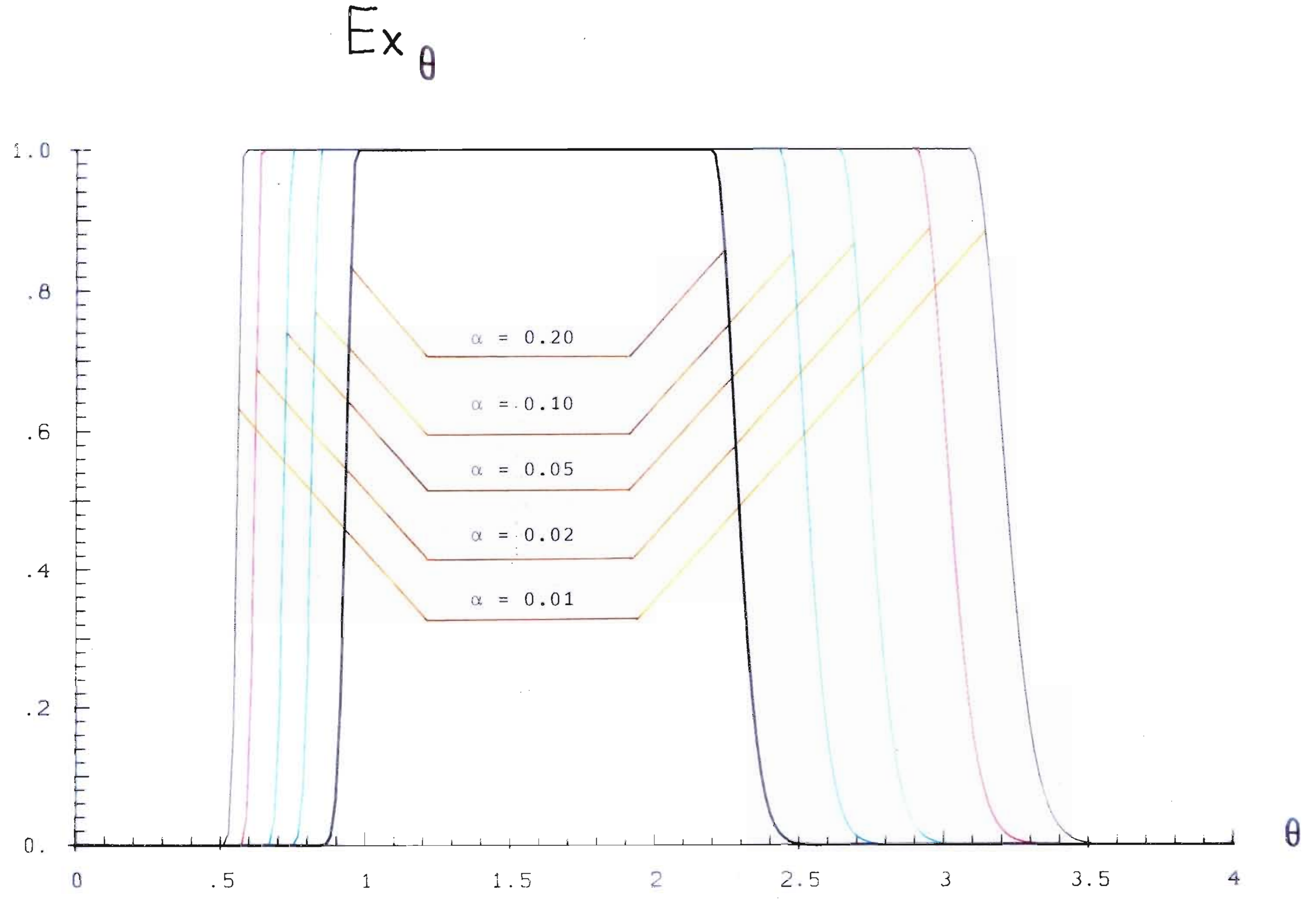
${}^\delta HPD_{1-\alpha}(g)$ HPD-Region based on g

$$A_\delta := \bigcup_{g \in \mathcal{D}_\delta} {}^\delta HPD_{1-\alpha}(g) \qquad \forall \delta \in (0;1]$$

$\Rightarrow (A_\delta ; \delta \in (0;1])$ nested family of subsets of Θ

Construction Lemma for Membership Functions:

$$\varphi(\theta) := \sup \{ \delta \cdot \mathbb{1}_{A_\delta}(\theta) : \delta \in [0;1] \} \qquad \forall \theta \in \Theta$$



Ex θ
1.0
.8
.6
.4
.2
0.
α = 0.20
α = 0.10
α = 0.05
α = 0.02
α = 0.01
0
.5
1
1.5
2
2.5
3
3.5
4
θ

PREDICTIVE DENSITIES

$X \sim f(\cdot|\theta)$, $\theta \in \Theta$ Stochastic Model

$\pi(\cdot)$ a-priori density

$(x_1, \cdots, x_n) = D$ data

$\Rightarrow$ $\pi(\cdot|D)$ a-posteriori density

$p(\cdot|D)$ predictive density

$$p(x|D) = \int_\Theta f(x|\theta) \cdot \pi(\theta|D)\, d\theta \quad \forall x \in M_X$$

FUZZY PREDICTIVE DENSITY

$p^*(\cdot \mid D^*)$

$$p^*(x \mid D^*) = \int_{\Theta} f(x \mid \theta) \odot \pi^*(\theta \mid D^*)\, d\theta \qquad \forall x \in M_X$$

$$\mathcal{D}_\delta := \{ g(\cdot) \text{ density on } \Theta : \underline{\pi}_\delta(\theta) \le g(\theta) \le \overline{\pi}_\delta(\theta) \ \forall \theta \in \Theta \}$$

$$a_\delta := \inf \left\{ \int_{\Theta} f(x \mid \theta)\, g(\theta)\, d\theta : g(\cdot) \in \mathcal{D}_\delta \right\}$$

$$\forall \delta \in (0; 1]$$

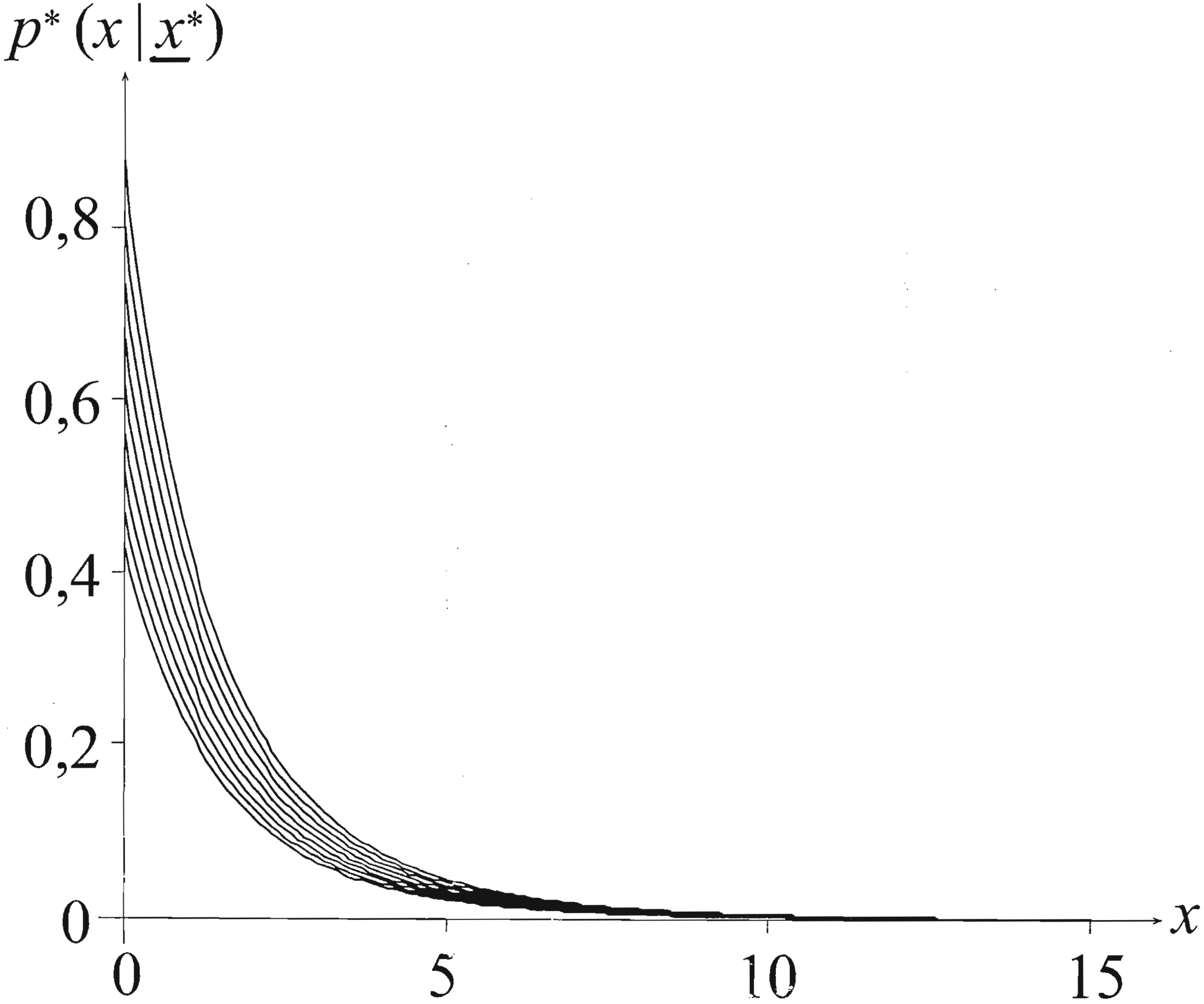
$$b_\delta := \sup \left\{ \int_{\Theta} f(x \mid \theta)\, g(\theta)\, d\theta : g(\cdot) \in \mathcal{D}_\delta \right\}$$

The nested family of intervals $[a_\delta; b_\delta]$ define a fuzzy number by the construction lemma:

$$\psi_x(y) = \sup\left\{\delta \cdot \mathbb{1}_{[a_\delta; b_\delta]}(y) : \delta \in [0;1]\right\} \quad \forall y \in \mathbb{R}$$

$$p^*(x|D^*) \triangleq \psi_x(\cdot)$$

For variable x this is a <u>fuzzy density</u>



$p^*(x\,|\,\underline{x}^*)$
0,8
0,6
0,4
0,2
0
0
5
10
15
x

SOFTWARE

- Some Programs

  C++, R

- Under Development:

  SAFD, ECSC

CONCLUSIONS

- Fuzziness can be described quantitatively
- Statistics based on fuzzy information is possible: Two different uncertainties
- Kolmogorov's probability concept has to be generalized
- Hybrid approach: Fuzzy and Stochastics

SOME REFERENCES

T. Ross et al. (Eds.): Fuzzy Logic and Probability Applications - Bridging the Gap, ASA and SIAM, Philadelphia, 2002

C. Borgelt et al. (Eds.): Combining Soft Computing and Statistical Methods in Data Analysis, Springer, Berlin, 2010

R. Viertl: Statistical Methods for Fuzzy Data, Wiley, Chichester, 2011