The design of blocked experiments when the average replication is very low

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Abstract

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However, recent work shows that
when there are blocks and the average replication is less than 2 then the best designs are far from obvious.

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If field operations have been primarily in one direction for a long time, then it is reasonable to divide the fields into blocks whose length runs along that direction.

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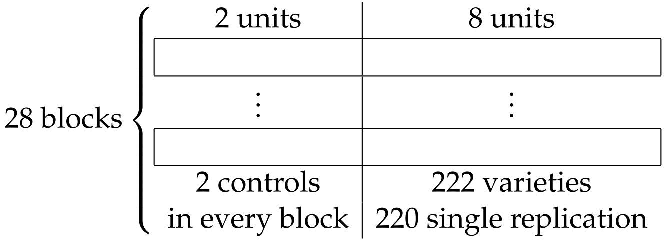

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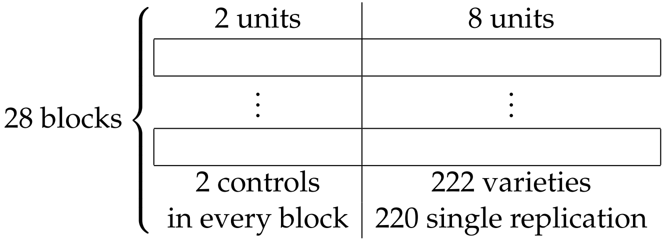
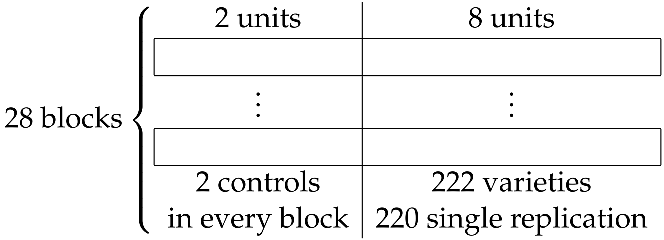


One extreme: 2 "controls" (among the test varieties) in every block.
Even more extreme: 2 uninteresting controls in each block.

Two possible designs for 224 varieties in 28 blocks of 10



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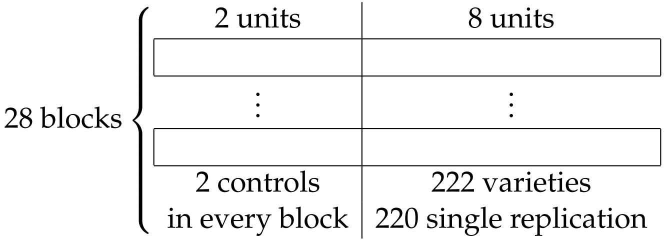

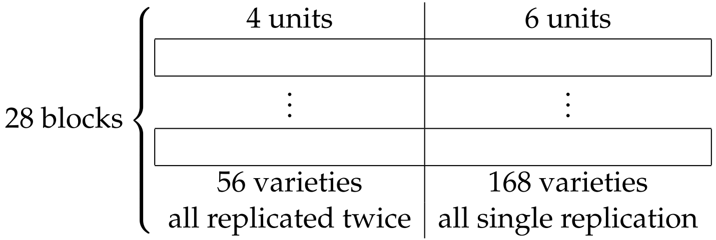

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We are given b blocks of size k. We are given v varieties. Assume that

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- Controls in every block.
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What makes a block design good?

A-optimal designs

We measure the response Y on the plot
with variety i in block D, and assume that

$$Y = \tau_i + \beta_D + \text{random noise},$$

where the random noise is $N(0, \sigma^2)$,
independently for each plot.

Put

$$V_{ij}\,\sigma^2 \quad = \quad \begin{array}{l}\text{variance of the best linear unbiased estimator}\\ \text{for } \tau_i - \tau_j;\end{array}$$

$$V_T = \sum_{i=1}^{v-1} \sum_{j=i+1}^{v} V_{ij} \quad \propto \quad \text{sum of variances of variety differences.}$$

A block design is A-optimal if it minimizes V_T.

Silly names just for this talk

Definition

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- a drone if it has replication 1;

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- a drone if it has replication 1;
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- a worker otherwise.

Is it better to put all the drones into one block (or a few blocks), or are they better distributed equally among all the blocks?

How should we distribute the drones?

Block A	Block B
n drones	m drones

If i is a drone in block A and j is a drone in block B then

$$V_{ij} = 2 + V_{AB},$$

where $V_{AB}\sigma^2$ is the variance of the estimator of the difference between the block effects of A and B in the design obtained by ignoring the drones.

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Then we have to remove m non-drones from block A,
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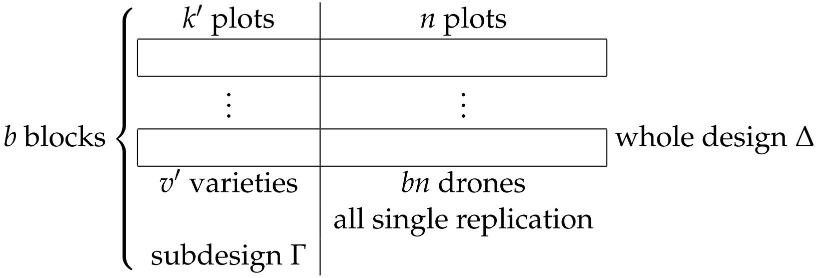
Then we have to remove m non-drones from block A,
and this increases the variances between these $n + m$ drones and the remaining $v - n - m$ varieties. This more than compensates for the original reduction in variance.

From now on, distribute drones as equally as possible



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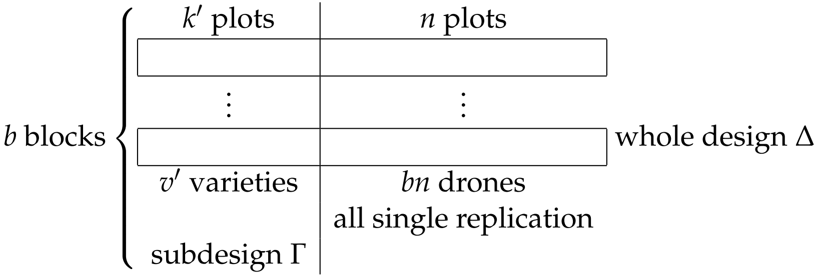

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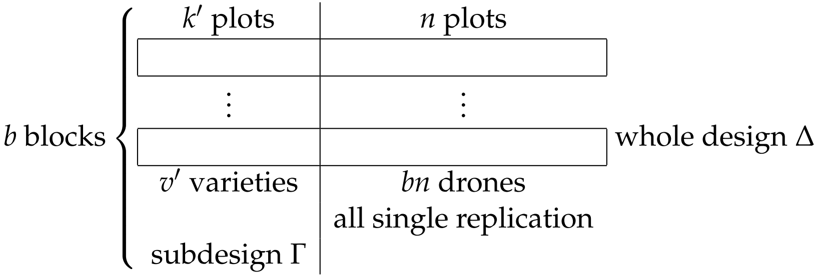

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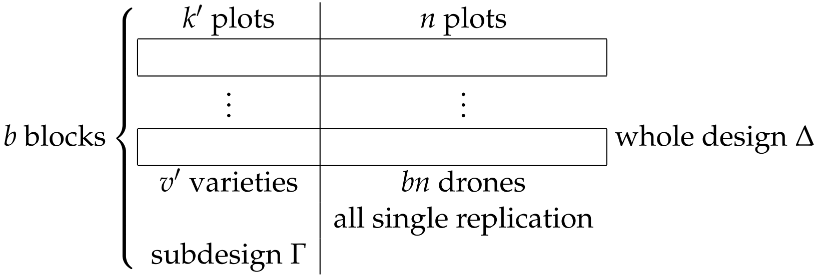

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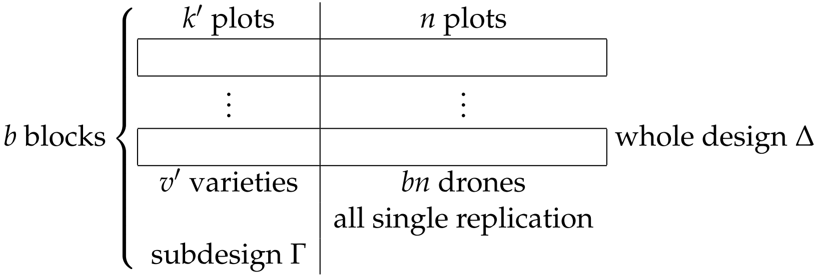

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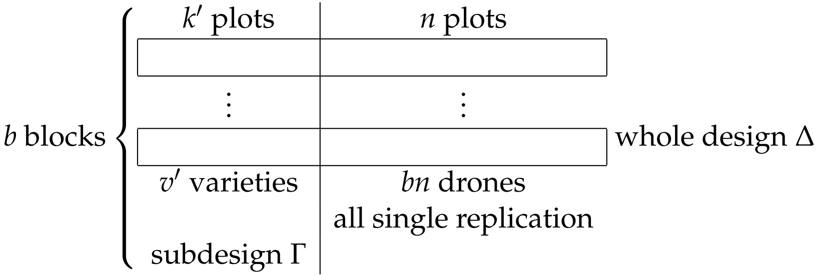

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$$n \geq n_0 = \left\lfloor \frac{2v - bk}{b} \right\rfloor \qquad k' \leq k_0 = k - n_0$$

Sum of the pairwise variances

Theorem (cf. Herzberg and Jarrett, 2007)

If there are n drones in each block of Δ,
and the core subdesign Γ has v' varieties in b blocks of size k'
then the sum of the variances of variety differences in Δ

$$= V_T(\Delta) = bn(bn + v' - 1) + V_T(\Gamma) + nV_{BT}(\Gamma) + n^2 V_B(\Gamma),$$

where

$$\begin{aligned} V_T(\Gamma) &= \text{the sum of the variances of variety differences in } \Gamma \\ V_B(\Gamma) &= \text{the sum of the variances of block differences in } \Gamma \\ V_{BT}(\Gamma) &= \text{the sum of the variances of (the estimators of) sums} \\ &\quad\ \text{of one variety effect and one block effect in } \Gamma. \end{aligned}$$

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If Γ is equi-replicate with replication r' then

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so $V_B(\Gamma)$ and $V_{BT}(\Gamma)$ are both increasing functions of $V_T(\Gamma)$.

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Consequence

For any given k', use the core subdesign Γ which minimizes $V_T(\Gamma)$.

Sum of variances in whole design if there are many drones

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Consequence

If v is large then n is large, so we need to focus on reducing $V_B(\Gamma)$,
so it may be best to increase the number of drones
and decrease k' (the size of blocks in the core subdesign Γ),
so that average replication within Γ is more than 2.

An example of this non-intuitive result

If there are $4(2+n)$ varieties in 4 blocks of size $4+n$, the design on the left is A-better than the design on the right if and only if $n < 50$.

1	2	3	4	n drones
1	2	5	6	n drones
3	6	7	8	n drones
4	5	7	8	n drones

1	2	3	$n+1$ drones
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1	3	4	$n+1$ drones
2	3	4	$n+1$ drones

Note that the core subdesign Γ on the right is
the dual of a balanced incomplete-block design because every
pair of blocks have the same number of varieties in common.

A definite result

Theorem

Suppose that we are given b blocks of size k, and v varieties.
For $i = 1, 2$, let design Δ_i have core subdesign Γ_i with block size k_i, where $k_1 > k_2$.
If Γ_1 is the dual of a balanced incomplete-block design then Δ_2 is worse than Δ_1 on the A criterion, no matter how big v is.

An example of the good result

If there are $4n + 6$ varieties in 4 blocks of size $3 + n$, the design on the left is A-better than the design on the right, for all values of n.

1	2	3	n drones
1	4	5	n drones
2	4	6	n drones
3	5	6	n drones

1	2	$n + 1$ drones
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Case 3. $k_0 \geq b-1$.

$$k_0 = k - \left\lfloor \frac{2v - bk}{b} \right\rfloor = \text{biggest space per block for non-drones.}$$

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Case 3. $k_0 \geq b-1$.

Case 4. $2 < k_0 < b-1$ (small k_0 but not Case 2).

$$k_0 = k - \left\lfloor \frac{2v - bk}{b} \right\rfloor = \text{biggest space per block for non-drones.}$$

Case 1. Only 2 blocks, of size k

Morgan and Jin (2007) showed that the A-optimal designs are those with $2n$ drones and q queen bees,
where $n = n_0 = v - k$ and $q = k' = k_0 = k - n_0 = 2k - v$.

1	2	3	4	...	q	A_1	A_2	A_3	...	A_n
1	2	3	4	...	q	B_1	B_2	B_3	...	B_n
queens						drones				

Case 1 continued. 3 blocks of size k

Using the nice theorem, RAB has shown that
the A-optimal designs are as follows when v is divisible by 3
(and presumably small changes deal with the other cases).
There are $3w$ workers and $3n$ drones,
where $3w = 3k - v$ and $n = n_0 = k - 2w$ and $k' = k_0 = 2w$.

1	2	4	5	...	$3w-2$	$3w-1$	A_1	A_2	A_3	...	A_n
1	3	4	6	...	$3w-2$	$3w$	B_1	B_2	B_3	...	B_n
2	3	5	6	...	$3w-1$	$3w$	C_1	C_2	C_3	...	C_n

w copies of design using all pairs from 3

drones

Case 2. $v = b(k-1)+1$

This is the maximum number of varieties that can be tested in b blocks of size k with all comparisons estimable.

Mandal, Shah and Sinha (1991), for $k = 2$,
and Bailey and Cameron (2013), for general block size,
showed that, no matter how many blocks there are,
the A-optimal design has the following form.

1	A_1	A_2	A_3	$\ldots$	A_{k-1}
1	B_1	B_2	B_3	$\ldots$	B_{k-1}
1	C_1	C_2	C_3	$\ldots$	C_{k-1}
1	D_1	D_2	D_3	$\ldots$	D_{k-1}
1	E_1	E_2	E_3	$\ldots$	E_{k-1}
1 queen	$v-1$ drones				

Case 2 continued. $v = b(k-1)$

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small k and b

1	2	A_1
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3	4	C_1
4	5	D_1
5	6	E_1
6	1	F_1

chain

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small k and b increase k if $b \geq 5$

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chain

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2	3	B_1	B_2
3	1	C_1	C_2
1	D_1	D_2	D_3
1	E_1	E_2	E_3
1	F_1	F_2	F_3

smaller chain

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small k and b | increase k if $b \geq 5$ | then increase b

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smaller chain

1	2	A_1	A_2
1	2	B_1	B_2
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1 queen

Youden and Connor (1953) had recommended chain designs.

Case 3. $k \geq k_0 \geq b - 1$

For simplicity, assume that b divides $2v$, so that

$$n_0 = \frac{2v - bk}{b} = \text{minimum number of drones per block.}$$

Then

$$\frac{b(2k - b + 1)}{2} \geq v \geq \frac{bk}{2} \geq \frac{b(b-1)}{2}.$$

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Let Γ_0 be the design for $b(b-1)/2$ varieties
replicated twice in b blocks of size $b-1$
in such a way that
there is one variety in common to each pair of blocks.
This is the dual of a balanced-incomplete block design
and so is A-optimal for these numbers.

Case 3 continued. $k \geq k_0 \geq b-1$

If $k_0 = s(b-1)$ then take Γ to be s copies of Γ_0.
The resulting whole design Δ is always A-optimal.

Case 3 continued. $k \geq k_0 \geq b - 1$

If $k_0 > b - 1$ but k_0 is not a multiple of $b - 1$,
then the following strategy seems likely to be good
(but Δ is not A-optimal when $b = k_0 = 4$ and v is very large).

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$n_0 =$ minimal number of drones per block.

Construction Method

1. *put n_0 drones in each block;*

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2. *put in one copy of Γ_0;*

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Construction Method

1. *put n_0 drones in each block;*
2. *put in one copy of Γ_0;*
3. *put in as many further copies of Γ_0 as possible;*

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Construction Method

1. *put n_0 drones in each block;*
2. *put in one copy of Γ_0;*
3. *put in as many further copies of Γ_0 as possible;*
4. *in any remaining space,*
   use a good design for workers with replication 2
   (so long as there is at least one copy of Γ_0,
   it probably doesn't make much difference which one is used).

Case 3. Example: $b = 8$ and $k = 15$ (so $60 \leq v \leq 92$)

60 varieties: all workers ($n_0 = 0$)

1	2	3	4	5	6	7	29	30	31	32	33	34	35	57
1	8	9	10	11	12	13	29	36	37	38	39	40	41	57
2	8	14	15	16	17	18	30	36	42	43	44	45	46	58
3	9	14	19	20	21	22	31	37	42	47	48	49	50	58
4	10	15	19	23	24	25	32	38	43	47	51	52	53	59
5	11	16	20	23	26	27	33	39	44	48	51	54	55	59
6	12	17	21	24	26	28	34	40	45	49	52	54	56	60
7	13	18	22	25	27	28	35	41	46	50	53	55	56	60
one copy of Γ_0							another copy of Γ_0							

Case 3. Example: $b = 8$ and $k = 15$ (so $60 \leq v \leq 92$)

76 varieties: 44 workers, 32 drones ($n_0 = 4$)

1	2	3	4	5	6	7	29	30	31	32	A_1	A_2	A_3	A_4
1	8	9	10	11	12	13	33	34	35	36	B_1	B_2	B_3	B_4
2	8	14	15	16	17	18	37	38	39	40	C_1	C_2	C_3	C_4
3	9	14	19	20	21	22	41	42	43	44	D_1	D_2	D_3	D_4
4	10	15	19	23	24	25	29	33	37	41	E_1	E_2	E_3	E_4
5	11	16	20	23	26	27	30	34	38	42	F_1	F_2	F_3	F_4
6	12	17	21	24	26	28	31	35	39	43	G_1	G_2	G_3	G_4
7	13	18	22	25	27	28	32	36	40	44	H_1	H_2	H_3	H_4
Γ_0							16 workers replication 2				drones			

Case 3. Example: $b = 8$ and $k = 15$ (so $60 \leq v \leq 92$)

92 varieties: 28 workers, 64 drones ($n_0 = 8$)

1	2	3	4	5	6	7	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8
1	8	9	10	11	12	13	B_1	B_2	B_3	B_4	B_5	B_6	B_7	B_8
2	8	14	15	16	17	18	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
3	9	14	19	20	21	22	D_1	D_2	D_3	D_4	D_5	D_6	D_7	D_8
4	10	15	19	23	24	25	E_1	E_2	E_3	E_4	E_5	E_6	E_7	E_8
5	11	16	20	23	26	27	F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8
6	12	17	21	24	26	28	G_1	G_2	G_3	G_4	G_5	G_6	G_7	G_8
7	13	18	22	25	27	28	H_1	H_2	H_3	H_4	H_5	H_6	H_7	H_8

Γ_0 | drones

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For various values of $k_i \leq k_0$,
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$$\begin{aligned} V_T(\Gamma_i) &= \text{the sum of the variances of variety differences in } \Gamma_i \\ V_B(\Gamma_i) &= \text{the sum of the variances of block differences in } \Gamma_i \\ V_{BT}(\Gamma_i) &= \text{the sum of the variances of sums of} \\ &\quad\ \text{one variety and one block in } \Gamma_i. \end{aligned}$$

If there are n_i drones in each block then, in the whole design Δ,

$$V_T(\Delta) = bn_i(bn_i + v'_i - 1) + V_T(\Gamma_i) + n_i V_{BT}(\Gamma_i) + n_i^2 V_B(\Gamma_i).$$

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Use this formula to find the core subdesign which gives the smallest $V_T(\Delta)$.
As the number of varieties increases, it becomes more important to choose Γ_i with a small value of $V_B(\Gamma_i)$.

Case 4 continued. $k_0 = 4 < b - 1$, $V_B(\Gamma_i) \div b(b-1)/2$

Best design for b blocks known to RAB

	Γ_1	Γ_2	Γ_3	Γ_4
k_i	2	3	3	4
	2 queens, both boring	2 queens, 2 workers (rep 2) $b-4$ drones	b workers rep 3	$2b$ workers rep 2
$b=6$	1	1^-	0.85	0.87
$b=7$	1	1^-	0.86	0.92
$b=8$	1	1^-	0.89	0.93
$b=9$	1	1^-	0.92	
$b=10$	1	1^-		
$b=11$	1	1^-		
$b=12$	1	1^-	0.98	
$b=13$	1	1^-	1	1.07
$b=14$	1	1^-		
$b=15$	1	1^-	1.01	1.08

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$b=11$	1	1^-		
$b=12$	1	1^-	0.98	
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As v increases, Γ_3 becomes better than Γ_4.

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$b = 10$	1	1^-		
$b = 11$	1	1^-		
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If $b \geq 14$, then, as v increases, Γ_1 and Γ_2 become better than Γ_3.

Case 4 continued. $2 < k_0 < b - 1$ when $b = 8$: $k_0 = 6$ so $v = 8k - 24$

$k = k_0 = 6$, and 24 varieties, all workers, all replicated twice.

A	1	2	3	4	5	6
B	7	8	9	10	11	12
C	1	7	13	14	15	16
D	2	8	17	18	19	20
E	3	9	13	17	21	22
F	4	10	14	18	23	24
G	5	11	15	19	21	23
H	6	12	16	20	22	24

(One worker for each pair of blocks
except for $\{A, B\}$, $\{C, D\}$, $\{E, F\}$ and $\{G, H\}$.)

Case 4 continued. $k = 5$ and $k = 6$ when $b = 8$: $k_0 = 5$ so $v = 8k - 20$

$k = 5$
20 varieties:
20 workers, no drones

1	2	3	4	5
6	7	8	9	10
1	11	12	13	14
2	6	15	16	17
3	7	11	18	19
4	8	12	15	20
5	9	13	16	18
10	14	17	19	20

$k = 6$
28 varieties:
20 workers, 8 drones

1	2	3	4	5	A_1
6	7	8	9	10	B_1
1	11	12	13	14	C_1
2	6	15	16	17	D_1
3	7	11	18	19	E_1
4	8	12	15	20	F_1
5	9	13	16	18	G_1
10	14	17	19	20	H_1

Case 4 continued. $k = 5$ and $k = 6$ when $b = 8$: $k_0 = 4$ so $v = 8k - 16$

$k = 5$

24 varieties:

16 workers, 8 drones

1	2	3	4	A_1
5	6	7	8	B_1
9	10	11	12	C_1
13	14	15	16	D_1
1	5	9	13	E_1
2	6	10	14	F_1
3	7	11	15	G_1
4	8	12	16	H_1

$k' = 4$

rep = 2

$k = 6$

32 varieties:

8 workers, 24 drones

1	2	4	A_1	A_2	A_3
2	3	5	B_1	B_2	B_3
3	4	6	C_1	C_2	C_3
4	5	7	D_1	D_2	D_3
5	6	8	E_1	E_2	E_3
6	7	1	F_1	F_2	F_3
7	8	2	G_1	G_2	G_3
8	1	3	H_1	H_2	H_3

$k' = 3$

rep 3

Case 4 continued. $k = 5$ and $k = 6$ when $b = 8$: $k_0 = 3$ so $v = 8k - 12$

$k = 5$
28 varieties:
12 workers, 16 drones

1	2	3	A_1	A_2
1	4	5	B_1	B_2
4	6	7	C_1	C_2
6	8	9	D_1	D_2
2	8	10	E_1	E_2
5	10	11	F_1	F_2
7	11	12	G_1	G_2
3	9	12	H_1	H_2

$k = 6$
36 varieties:
12 workers, 24 drones

1	2	3	A_1	A_2	A_3
1	4	5	B_1	B_2	B_3
4	6	7	C_1	C_2	C_3
6	8	9	D_1	D_2	D_3
2	8	10	E_1	E_2	E_3
5	10	11	F_1	F_2	F_3
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Health Warnings

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This is work is progress, not a finished project.